School Geometry

Anthony G. O'Farrell
Mathematics Department
NUI, Maynooth
Co. Kildare
email: admin@maths.nuim.ie

April 24, 2008

1 Levels of Sophistication

In the following, Geometry refers to plane geometry.

In designing the programme in Geometry for schools, it is essential to ensure the logical coherence of the system that is used. This is not something that can be taken for granted, and in fact, things have gone wrong in the past in Ireland[8].

We distinguish three levels:

Level 1: The fully-rigorous level, likely to be intelligible only to professional mathematicians and advanced third- and fourth-level students.

Level 2: The semiformal level, suitable for digestion by many children from (roughly) the age of 14 and upwards.

Level 3: The informal level, suitable for younger children.

In schools, one should begin with Level 3, developing the pupils' geometrical intuition and teaching them geometrical language, such as the names of various shapes. One should progress to geometrical problem-solving, addressing concrete questions. At an appropriate point (Secondary Year 3 seems to be favoured), one should introduce the semiformal Level 2, involving such things as definitions, theorems, and proofs. At no stage should the material of the fully-rigorous Level 1 be used in school. Its purpose is to ensure that the semiformal level is gounded on a sound system, and to act as a guide to the construction of Level 2.

2 Relationship of the Levels

2.1 Levels 2 & 3

The undefined terms, defined terms, and axioms of Level 2 should be based on experience at Level 3. Nothing should be asserted at Level 3 that is not (semi-formally) provable at Level 2.

2.2 Levels 1 & 2

Level 2 should be constructed from Level 1 by simplifications, such as the elimination of technical terminology and notation needed for fine distinctions and a generally more relaxed approach to the detail of proofs.

Every statement and proof in Level 2 should have a completely-rigorous counterpart in Level 1.

3 Alternatives for Level 1

There are many possible versions of Level 1. All versions should, in the end, provide derivations of precisely the same list of geometrical statements. But different choices will affect the logical sequence in Level 2, so it is essential to pick one specific system, as long as we are operating a uniform national system of examination and assessment.

To elaborate, let A and B be two versions of Level 1. Each will have undefined terms (i.e. undefined words), defined terms, definitions, axioms and theorems. Let

$$\begin{aligned} A_u &= \text{the set of undefined terms of A,} \\ A_w &= \text{the set of defined terms of A,} \\ A_d &= \text{the set of definions A,} \\ A_a &= \text{the set of axioms of A,} \\ A_t &= \text{the set of theorems of A.} \end{aligned}$$

Note that A_u and A_w are sets of words, whereas A_d, A_a and A_t are sets of statements. Let B_u, B_w, B_d, B_a and B_t be, respectively, the sets of undefined terms, defined terms, definitions, axioms and theorems of system B. Then, whereas the entire collection of terms will be the same in both theories:

$$A_u \cup A_w = B_u \cup B_w,$$

(apart from trivial changes involving replacement of a word by a synonym, and assuming definitions are added for any terms of A that do not appear at all in B, and vice-versa.) it may happen that terms that are undefined in A are defined in B, and vice-versa. In symbols, we may have

$$A_u \cap B_w \neq \emptyset \neq B_u \cap A_w.$$

Even more significantly, although the entire collection of statements

$$A_d \cup A_a \cup A_t = B_d \cup B_a \cup B_t$$

will be the same in both, it may happen that assumptions (axioms) in A become theorems in B, and vice-versa, so we may have

$$A_a \cap B_t \neq \emptyset.$$

For instance, in A one might have Playfair's Axiom of parallels

(Through any point in the plane, there is at most one straight line parallel to a given straight line.)

as an axiom, and get the statement that the angle-sum in each triangle is equal to two right angles as a theorem, and in B their rôles could be reversed. Obviously, a Level 2 programme based on B must differ markedly from one based on A.

4 How to Choose?

At the NCCA Junior Cycle Course Committee for Mathematics, I was asked to recommend a suitable Level 1, on which to base our programmes. How does one address this question?

4.1 Target

Let us denote the union of all the provable propositions[1] in system A by

$$A_p = A_d \cup A_a \cup A_t.$$

Considering the entire body of (Hilbert-)Euclidean synthetic Geometry, and coordinate geometry, the chosen system must deliver an A_p that includes the standard theorems of Geometry and the standard properties of the real number system.

4.2 Types of System

In the literature, there are two broad categories of geometrical system. The first treats "purely synthetic geometry" — Geometry without real numbers — separately, and either requires a separate, self-contained theory of the real numbers, or actually constructs the real numbers from geometrical objects. We will refer to all such systems as systems of *Hilbert type.* the second type begins with (or takes for granted) a self-contained theory of the real numbers, and uses real numbers in the axiomatization of Geometry. We shall refer to such systems as of *Birkhoff type.*

The first successful example of a system of Hilbert type was Hilbert's [6]. Hilbert added precision and additional axioms to Euclid's system, in order to repair its deficiencies and allow the rigorous deduction of all Euclid's theorems. (The main lacunae in Euclid had to do with congruence, betweenness, and completeness.) Hilbert's system has two axioms of connection, five axioms of order, an axiom of parallels

(Through any point in the plane, there is one and only one straight line parallel to a given straight line.)

six axioms of congruence, an Archimedean axiom, and an axiom of completeness.

Variations on Hilbert's system may be found described in many books designed for use in university-level courses for prospective teachers, such as Greenberg [5] or Cederberg [4]. Typically, one has about 15 axioms.

[1]Purists will have noted that, logically, the axioms and definitions are theorems with very, very simple proofs, so that $A_p = A_t$. I have been quietly assuming that a theorem is a statement provable in the theory *that is not already a definition or axiom.*

The first successful example of Birkhoff type was David Birkhoff's [2]. Taking the system of real numbers as given, he gave just four axioms. These axioms are about line measure, incidence, angle measure, and similar triangles. (In fairness to Hilbert, some of Birkhoff's axioms assert more than one simple statement.) The axioms of line measure and angle measure are usually called The Ruler and Protractor Axioms. This system, described fully in his paper in the Annals of Mathematics [2], was translated to Level 2 in the textbook of Birkhoff and Beatley [3], resulting in an account with five axioms. Variations on this are in common use in popular texts in the USA, such as [7].

4.3 Comparison

In discussions between the Heads of the National University of Ireland Mathematics schools, it was agreed that a system of the Birkhoff type has significant advantages.

- One advantage is that a Level 1 of Birkhoff type buries the difficulties associated to completeness in the underlying system of real numbers, and there is no explicit reference to completeness in the geometrical axioms. (Without completeness, a geometrical theory will not be *categorical*, i.e. there will be several different Euclidean geometries, in some of which, for instance, a line through the centre of a circle may fail to meet the circumference, and various constructions in Euclid may fail to attain their object.)

- Another advantage is that the use of real numbers makes the geometry simpler.

- Pupils may not, in fact, really understand much about the real number system, but they are familiar with it, on an intuitive level.

- The transition to coordinate geometry[2] is simpler.

Among systems of Birkhoff type, Birkhoff's own does not use an axiom of parallels. As a result of attempts to prove Euclid's Fifth Postulate, beginning seriously with the work of Saccheri (1667-1733), we know that there are

[2]By the way, it does not appear to be widely recognized that traditional coordinate geometry rests on the parallel postulate, via Pythagoras' Theorem.

various other plausible-looking axioms which, taken together with "uncontroversial" axioms of incidence, betweenness, and congruence, may be used to prove the axiom of parallels as a theorem. For instance, it is enough to assume that there exists some rectangle (i.e. a quadrilateral having four right angles). Another possibility is to assume that, given a triangle, it is possible to find a similar triangle having any given segment as one side. This is the axiom that Birkhoff uses.

Given the fact that many teachers are used to something along the lines of Playfair's or Hilbert's version, it is probably preferable to use a system that sticks more closely to Euclid on this one.

4.4 Conclusion

So one is looking for a Level 1 description of Birkhoff type, that employs an axiom of parallels. Ideally, this description should be already available in print. Necessarily, it should be thoroughly professional and complete. In particular, the account should include fully-articulated and rigorous proofs of all the theorems we might include in Junior or Senior Certificate programmes now or in the future. Without this, people who wish to construct Level 2 textbooks will have insufficient guidance.

5 Recommendation

There do not appear to be many choices available, in fact. Accounts written by experts tend to omit the detailed working-through of the theorems (because it is well-known that these can be proven once a small subset is proven), and so do not provide a reliable guide to those who will compose a simplified version at Level 2 for school use.

One book, by an Irish author, stands out. It is P.D. Barry's *Geometry and Trigonometry* [1]. Chapters 2 to 5 present a careful and complete Level 1 account, including rigorous proofs of all the theorems currently listed in the programme. There is a good deal more in the book, that builds on this foundation, and could inform Level 2 accounts of other areas of the programme.

Barry's system has the primitive undefined terms *plane, point, line,* $<_l$ *(precedes on a line), (open) half-plane, distance, and degree-measure*, and seven axioms:

A_1: about incidence,

A_2: about order,

A_3: about plane separation,

A_4: about distance,

A_5: about degree measure,

A_6: about congruence of triangles (SAS),

A_7: about parallels (Playfair, in fact).

I recommend the adoption of Barry's system as the Level 1 basis for Geometry in our schools.

It is to be emphasized again that this is *not* a recommendation to teach the exact content of Chapters 2–5 of Barry's book in schools, nor is it even a recommendation that all Maths teachers should read Barry's book. The book is aimed at a readership of university-level students. It must be converted to a Level 2 account by suitable simplification, before school use. Any proposed conversion should be vetted by experts, to validate its conformity with the Level 1 version. Also, before any Level 2 account is used with students, they should first have a thorough preparation[3] at the informal Level 3.

The simplifications required to convert Barry's account to a Level 2 account would certainly include informal, as opposed to formal, treatment of the order of points on lines, and the separation of the plane into two pieces by lines, so that the axioms A_2 and A_3 would have no counterparts at Level 2.

References

[1] Patrick D. Barry. *Geometry with Trigonometry.* Horwood. Chichester. 2001. ISBN 1-898563-69-1.

[3] Barry's Chapter 1 (especially Section 1.5) provides some useful guidance as to which concepts need to be familiar to students at an informal level, before they proceed to Level 2.

[2] George D. Birkhoff. *A set of postulates for plane geometry, based on scale and protractor.* Annals of Math. **3**3 (1932) 329-45.

[3] George D. Birkhoff and Ralph Beatley. *Basic geometry, 3rd Ed.* Chelsea. New York. 1959.

[4] Judith N. Cederberg. *A Course in Modern Geometries.* Springer. New York. 1989.

[5] Marvin J. Greenberg. *Euclidean and Non-Euclidean Geometries, 3rd Ed.* W.H. Freeman. New York. 1993.

[6] David Hilbert. *Foundations of Geometry, 3rd Ed.* (translated by E.J. Townsend). Open Court. La Salle, Illinois. 1938

[7] Harold R. Jacobs. *Geometry: Seeing, Doing, Understanding, 3rd Ed.* W.H. Freeman. New York. 2003.

[8] Susan M.C. Mac Donald. *The Paradigm Shift from Euclid to a composite System of Geometry in Intermediate Certificate Mathematics in Ireland, 1966-1973.* (Two volumes, 355+333 pp.) Ph.D. Thesis, NUI, Maynooth. 2007.