Reversibility

Anthony G. O'Farrell

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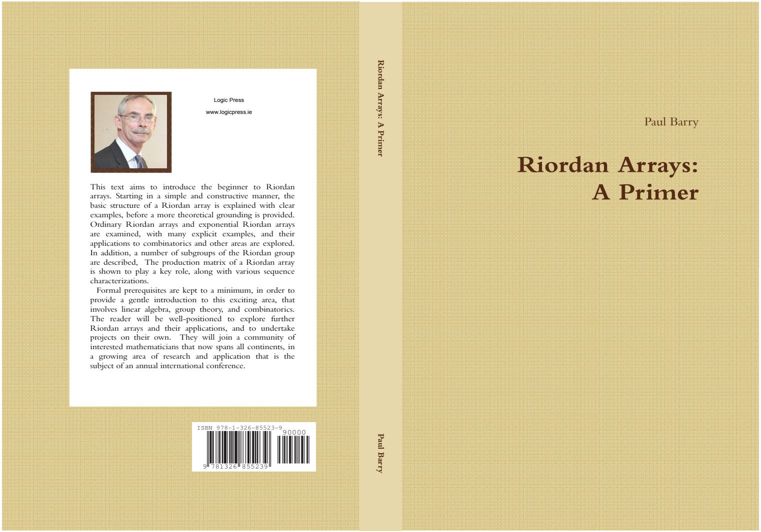

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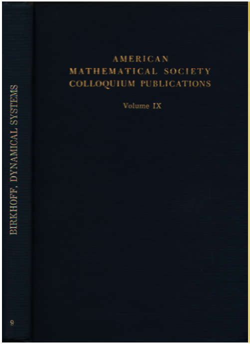
1. Origins in Dynamics
2. Reversibility in General
3. The Basic Questions
   - Questions
   - Progress to date
   - Subgroups and Quotients
   - Dihedral groups
   - Full Permutation Groups
4. Connections to other areas and problems
   - A Strategy
   - Poincaré's problem
   - Two-valued reflection
   - De Paepe's problem
   - $GL(2,\mathbb{Z})$
   - $f(x)+g(y)$
5. Open Questions

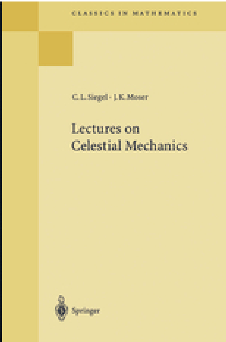
History: Laplace, G. D. Birkhoff, Siegel-Moser, Kolmogoroff, Arnold, (Smale) Devaney.

Birkhoff



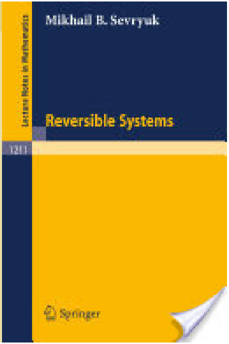
1927

Siegel+Moser



1956-1971

Sevryuk.



1986

The harmonic oscillator has the Hamiltonian

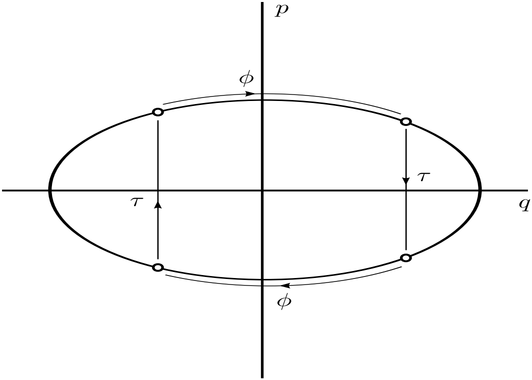
$$H : \begin{cases} \mathbb{R}^2 & \to & \mathbb{R}, \\ (q,p) & \mapsto & \dfrac{p^2}{2m} + \kappa q^2, \end{cases} \tag{1}$$

where q denotes the horizontal displacement of the bob from its equilibrium position $q = 0$, p denotes momentum, m denotes the mass of the bob, and κ is a constant.

The motion is determined by the first-order system

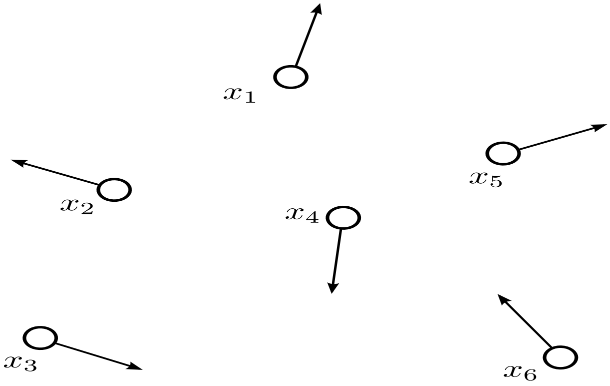
$$\begin{cases} \dfrac{dq}{dt} & = & \dfrac{\partial H}{\partial p}, \\ \dfrac{dp}{dt} & = & -\dfrac{\partial H}{\partial q}. \end{cases} \tag{2}$$

Obviously, $H(p,q)$ is constant along trajectories, so the trajectories are the concentric ellipses $H(p,q) = E$, for constant $E \geq 0$.



The Harmonic Oscillator

$$\tau \circ \phi \circ \tau \circ \phi = \mathbf{1}.$$

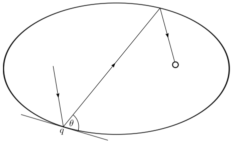


The *n*-body problem

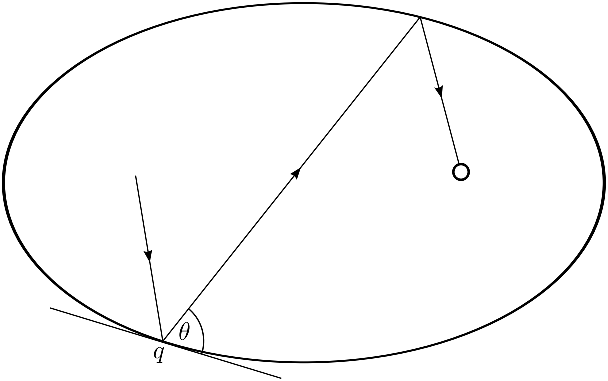
$H(p,q) := K(p) + V(q)$ has $H(-p,q) = H(p,q)$. Let ϕ be the time-one step, and $\tau(p,q) = (-p,q)$. Then again:

$$\tau \circ \phi \circ \tau \circ \phi = \mathbf{1}.$$

A discrete system:
Consider billiards on an arbitrary smoothly-bounded strictly-convex table without pockets.



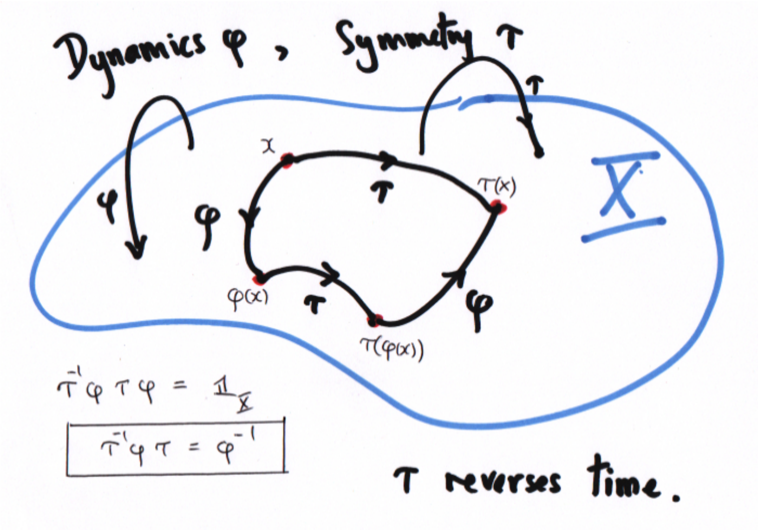
Let Γ denote the boundary.
We may parametrise the set of states at which the ball leaves the cushion by two parameters q and θ, where $q \in \Gamma$ is the point it leaves from, and θ is the angle between the line it departs on and the tangent to Γ in the counterclockwise direction.
Thus the state space is $X = \Gamma \times (0, \pi)$, and the dynamical step $\phi : X \to X$ is the map that takes (q_0, θ_0) to (q_1, θ_1), where (q_1, θ_1) parametrises the state that results one bounce after the state parametrised by (q_0, θ_0).



(Discrete) Billiards

Let $\tau(q,\theta) := (q, \pi - \theta)$. Then $\tau \circ \tau = \mathbf{1}$, and

$$\tau \circ \phi \circ \tau \circ \phi = \mathbf{1}.$$



Time-reversing symmetry τ

Reversibility in General Groups G

Denote conjugate $g^{-1}fg$ by f^g.

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products of n elements of a subset $A \subset G$:
$A^n := \{\tau_1 \cdots \tau_n : \tau_i \in A\}$.

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Lemma

(1) $I^2 \subset R$. (strongly-reversible elements)
(2) $R_f^2(G) \subset C_f(G) := \{g \in G : gf = fg\}$.

Theorem (Basic Theorem)

Let G be a group and $f, g \in G$. Then the following three conditions are equivalent:

1. *(1) $g \in R_f(G)$;*
2. *(2) there exists $h \in G$ with $g^2 = h^2$ and $f = g^{-1}h$;*
3. *(3) there exist $h \in G$ such that $f = gh$ and $f^{-1} = hg$.*

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Corollary

Let G be a group and $f \in G$. Then the following three conditions are equivalent:

1. *(1) $f \in R(G)$;*
2. *(2) there exist $g, h \in G$ with $g^2 = h^2$ and $f = g^{-1}h$;*
3. *(3) there exist $g, h \in G$ such that $f = gh$ and $f^{-1} = hg$.*

The Basic Questions

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- Describe I^∞.
- Describe R^∞.
- Is $I^n = I^\infty$ for some n?

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- Is $I^n = I^\infty$ for some n?
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The Basic Questions

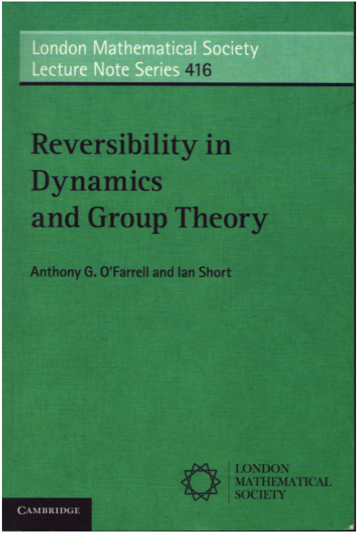
In each group, G, we ask:

- Which f are reversible in G?
- Which h reverse a given f?
- Describe I^∞.
- Describe R^∞.
- Is $I^n = I^\infty$ for some n?
- Is $R^n = R^\infty$ for some n?
- Does every nonempty R_g have an element of finite order? If so, what orders occur?

Results so Far

Basic results are to hand for:

- Finite groups
- Classical matrix groups over fields
- Isometry groups of constant-curvature geometries
- Homeomorphism groups of line and circle.
- Diffeo($\mathbb{R}$).
- Formal power-series groups in one and several variables
- $\mathrm{GL}(2,\mathbb{Z})$ and some relatives.
- The group of biholomorphic germs in one variable.
- Groups of piecewise-linear maps.
- Groups of area-preserving maps.
- Groups of polynomial maps.


London Mathematical Society
Lecture Note Series 416
Reversibility in Dynamics and Group Theory
Anthony G. O'Farrell and Ian Short
LONDON MATHEMATICAL SOCIETY
CAMBRIDGE

Lemma

Suppose that $\phi : G \to H$ is a group homomorphism. Then

$$\phi(I(G)) \subset I(H), \quad \phi(R(G)) \subset R(H).$$

Corollary

Let $\phi : G \to H$ be a homomorphism. Then $R(G) \subset \phi^{-1}(R(H))$.

Thus if we are investigating reversibility in a group G, and encounter difficulties, we should first investigate reversibility in its quotient groups. Since these are "smaller", the problem should be easier in them.
Also, it may be advantageous to consider reversibility in larger groups K in which G embeds. If $G \leq K$, then it is easier for an element $g \in G$ to belong to $R(K)$ than for it to belong to $R(G)$.

The dihedral groups are those generated by two involutions.
Concretely, D_n is the group of isometries of the regular n-gon, and D_∞ is the group of isometries of $\mathbb{Z}$.
If G is a dihedral group, then $I^2(G) = G$.

Consider the full permutation group $S(X)$ on an arbitrary set X.
For each $a \in X$ and $\tau \in S(X)$, the (two-sided) *orbit* of a under τ is the set

$$\mathrm{Orb}(\tau, a) = \{\tau^n(a) : n \in \mathbb{Z}\}.$$

τ is reversible in $S(X)$ if each restriction $\tau|\mathrm{Orb}(\tau, a)$ is reversible in $S(\mathrm{Orb}(\tau, a))$. But this always holds!

Proposition

For each set X, each element of $G = S(X)$ is strongly reversible, i.e. $I^2(G) = R(G) = G$.

Each group G may be regarded as a group of bijections. The proposition shows that as "unstructured mappings", all bijections are strongly reversible. This means that the interesting aspects of reversibility are not a matter of set theory: the interesting questions concern groups of maps that have additional structure of some kind.

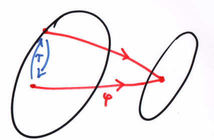
Take, for instance, the homeomorphism group $G = \mathrm{Homeo}(X)$, where X is a topological space, and the group consists of all the homeomorphisms of X onto itself. On each orbit of a homeomorphism τ, the homeomorphism may be factored as a product of two involutions. But even if these could be taken to be continuous on each orbit, there would typically be many choices of the factorization on each orbit, and τ would only be reversible in G if these choices could be made in such a way that the factors on the orbits patched together continuously.

Even for $X = \mathbb{R}$ it is impossible to factor each homeomorphism as the product of two continuous involutions.

Reversibility arises in:

- dynamics. Physics: Henon map. Taylor-Chirikoff map.
- group theory
- 1-D complex analysis: two-valued reflections, quadratic correspondences, conformal maps
- *n*-D complex analysis: biholomorphic invariants of curves and surfaces
- complex polynomial approximation
- real-variable approximation in two variables
- functional equations
- number theory: quadratic forms, $\mathrm{GL}(2,\mathbb{Z})$, real quadratic fields
- hyperbolic geometry: geodesics
- topology: toral automorphisms, foliations of Poincaré 3-manifolds

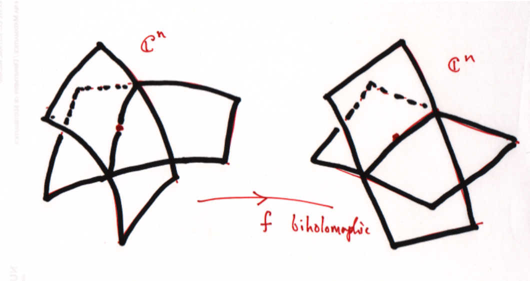
Useful Ideas:



- two-one map ↔ involution
- two involutions → reversible map (and dihedral group)
- reversible map → dynamical system (‘hidden’)
- dynamical system → toolkit

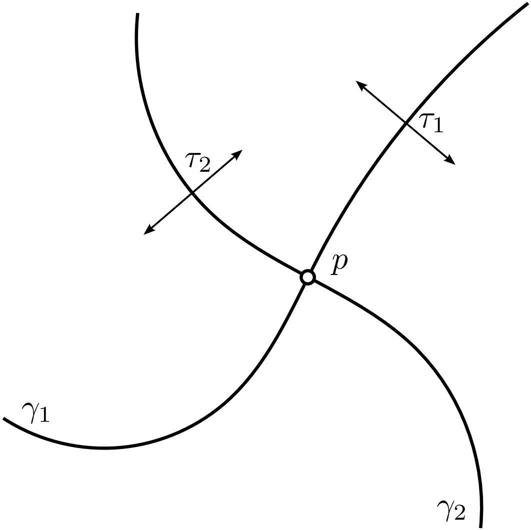
The toolkit includes normal forms, embedding in a flow, invariant measures, and ergodic theory.

Example: Poincaré problem: Classify (germs at a point of) sets $X \subset \mathbb{C}^n$ under (locally) biholomorphic change of variables:



Case 1.1: Real-analytic arc: Just one class.

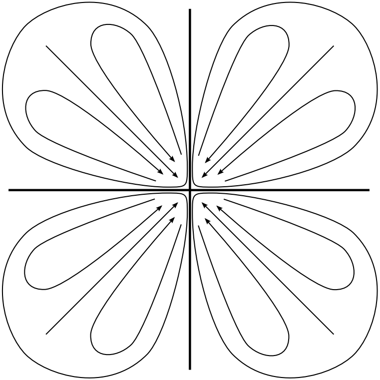
Case 1.2: Two arcs.



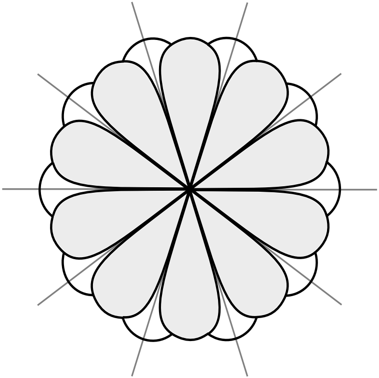
$$f(z) := \tau_1 \circ \tau_2(z) = mz + a_2 z^2 + \cdots .$$

(Arnol'd)

Tangent arcs: $f(z) := z + az^{p+1} + \cdots$.



(Écalle, Voronin, Nakai, Ahern+Gong, Ahern+OF)



Leau Flower

Case 2.1: Real-analytic surface:
Bishop, Moser-Webster.
Case 2.1.2: 'disk' with complex tangent.

$$z_2 = F(z_1, \overline{z_1}) = q(z_1, \overline{z_1}) + O(|z_1|^3)$$

$$z_2 = \gamma z_1^2 + z_1\overline{z_1} + \gamma\overline{z_1}^2 + O(|z_1|^3)$$

Complexification:

$$z_2 = F(z_1, w_1), \quad w_2 = \bar{F}(w_1, z_1).$$

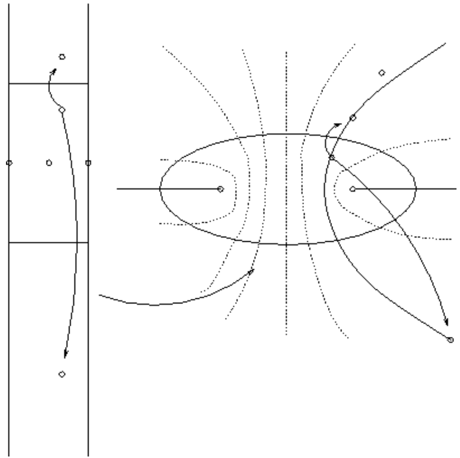
— a 2-D subvariety $\hat{X}$ of $\mathbb{C}^4$, which has locally two-to-one projections under the maps

$$\pi_1(z_1, z_2, w_1, w_2) \mapsto (z_1, z_2)$$

$$\pi_1(z_1, z_2, w_1, w_2) \mapsto (w_1, w_2).$$

This gives us two involutions on $\hat{X}$, hence on $\mathbb{C}^2$.

Example: Two-valued reflection. Webster.



Reflection in an Ellipse

Example: De Paepe's problem (1984): $f : \mathbb{C} \to \mathbb{C}$. Are there polynomials $p_n(t)$ such that

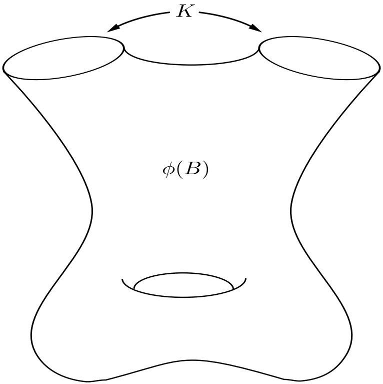
$$p_n(z^2, \bar{z}^2 + \bar{z}^3) \to f(z)$$

on small disks $\{|z| \le r\}$?

OF+Sanabria (1999): No.

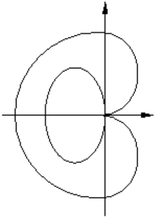
This comes down to polynomial convexity of an associated topological disk in $\mathbb{C}^2$, namely

$$X := \{(z^2, \bar{z}^2 + \bar{z}^3) : |z| \le r\}.$$



Polynomial hull

Holomorphic involutions that swap preimages of z^2 and $z^2 + z^3$ give biholomorphic germ $z + z^2 + \cdots$. Embedding this in a flow, and *complexifying the time* gives a map of the lunula to $\mathbb{C}^2$ that attaches a



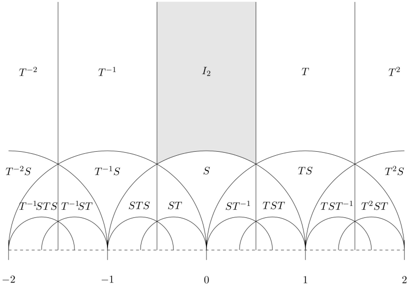
variety to X.

Example: $\mathrm{GL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1 \right\}$.
All elliptic and parabolic elements are reversible, and the only reversibles with determinant -1 are involutions, so the main focus is on the hyperbolic elements of $\mathrm{SL}(2,\mathbb{Z})$.
Consider $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $a = d$, then g is reversible in $\mathrm{GL}(2,\mathbb{Z})$. If $a \neq d$, then g is reversible if and only if an associated binary integral quadratic form $Ax^2 + Bxy + Cy^2$ represents ± 1.
If $a = d$, then g is reversible in $\mathrm{SL}(2,\mathbb{Z})$ if and only if an associated Pell equation $x^2 - ey^2 = -1$ has a solution. If $a \neq d$, then g is reversible in $\mathrm{SL}(2,\mathbb{Z})$ if and only if the quadratic form represents -1.

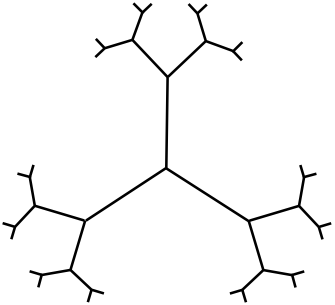
There are various objects associated to hyperbolic $g \in \mathrm{GL}(2, \mathbb{Z})$:

- a toral automorphisms
- a mapping class on the torus
- a linear fractional transformation fixing $\mathbb{R}$
- a geodesic on the modular surface
- a cyclically-reduced word in $C_2 * C_3$
- Foliations of a Poincaré 3-manifold
- an ideal in the maximal order of a real quadratic field
- an automorphism of the Markov tree

So reversibility of g in GL (or SL, or PSL) has many facets.

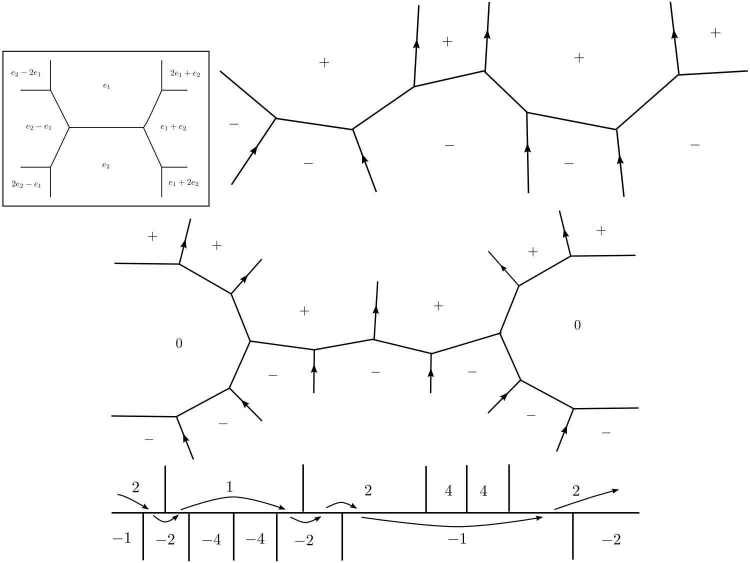


$$\frac{az+b}{cz+d}$$

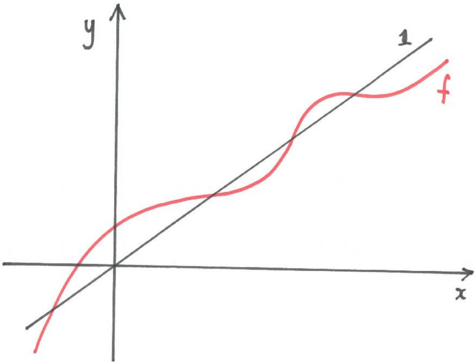


Markov tree

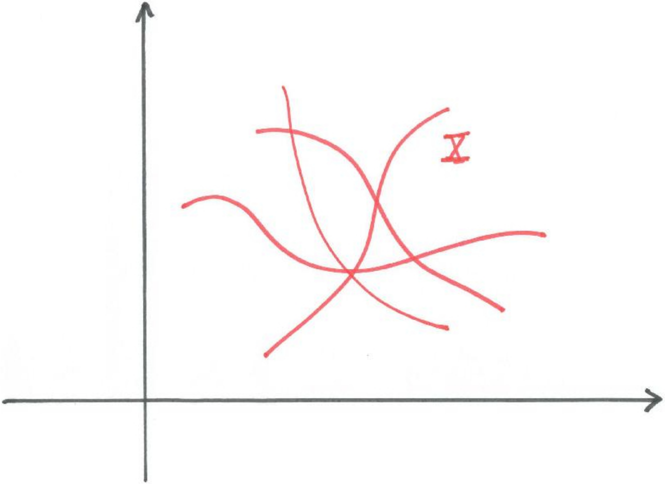
Conway's Topograph: River, Lakes, Meadows:



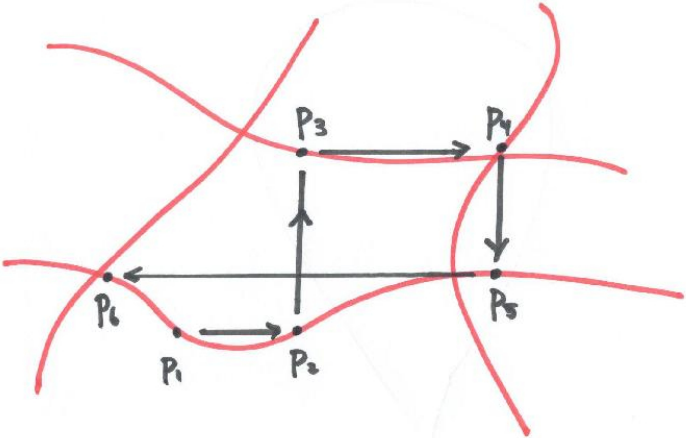
Example: When can you represent, or even approximate, a function of two real variables $h(x, y)$ by sums $f(x) + g(y)$?
— arises in nomography, functional equations, numerical analysis, economics.
E.g. the functional equation: $\alpha(f(x)) = \alpha(x) + g(x)$, $\forall x \in \mathbb{R}$ and



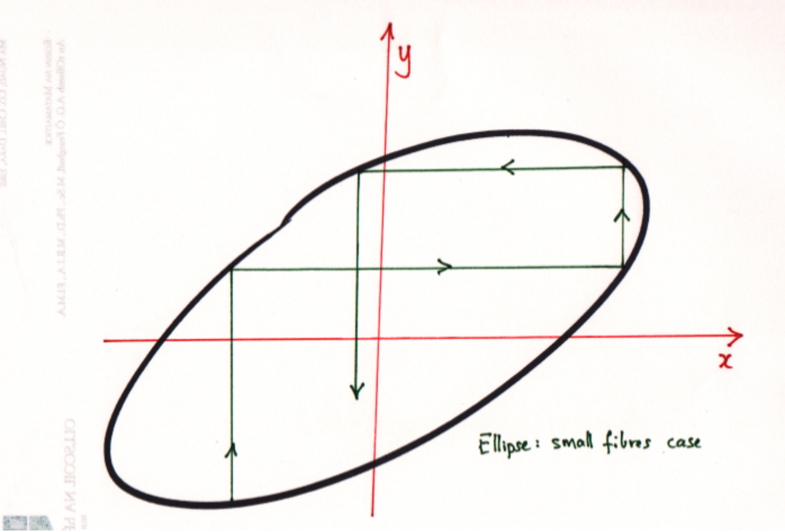
graph $\cup$ diagonal



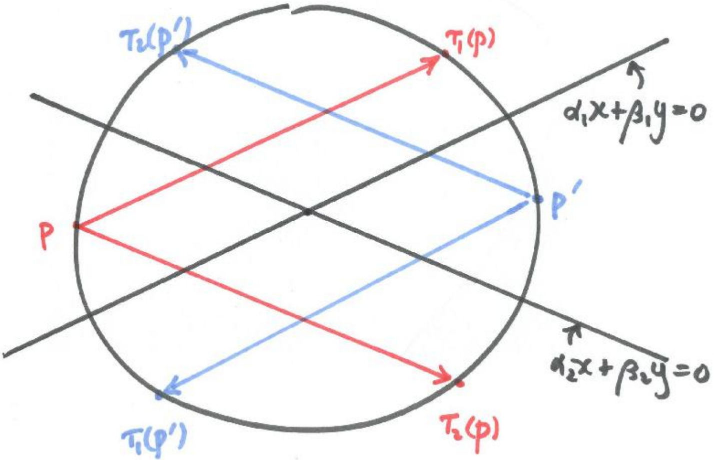
compact set in $\mathbb{R}^2$



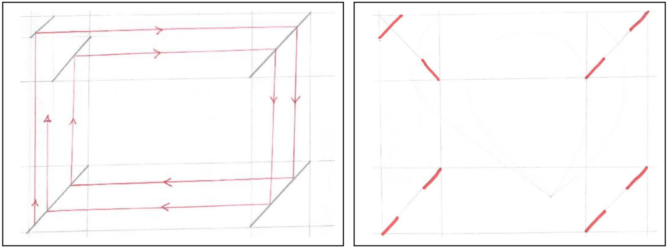
Lightning bolt



Bolt in Ellipse: small fibres



Ellipse transformed to circle



pathology department

Open Questions

- Your favourite group.
- See Chapter Notes in O'F+Short book
- Formal series over fields of nonzero characteristic, including the Nottingham group.
- Formal series and biholomorphic germs in two or more variables.

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Factoring formal maps into reversible or involutive factors ☆

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ABSTRACT

An element g of a group is called *reversible* if it is conjugate in the group to its inverse. An element is an *involution* if it is equal to its inverse. This paper is about factoring elements as products of reversibles in the group $\mathfrak{G}_n$ of formal maps of $(\mathbb{C}^n, 0)$, i.e. formally-invertible n-tuples of formal power series in n variables, with complex coefficients. The case $n = 1$ was already understood [25]. Each product F of reversibles has linear part $L(F)$ of determinant ± 1. The main results are that for $n \geqslant 2$ each map F with $\det(L(F)) = \pm 1$ is the product of $2 + 3 \cdot \text{ceiling}(\log_2 n)$ reversibles, and may also be factored as the product of $9 + 6 \cdot \text{ceiling}(\log_2 n)$ involutions (where the ceiling of x is the smallest integer $\geqslant x$).

formal maps of $\mathbb{C}^2$

Higher-dimensional Riordan groups:
For an integral domain A and $d \in \mathbb{N}$, let $\mathcal{F}$ be the formal power series ring $A[[x]] := A[[x_1, \ldots, x_d]]$, and $\mathcal{G} \subset x_1\mathcal{F} + \cdots + x_d\mathcal{F}$ be the group of formally-invertible elements. Then we can form two 'Riordan groups': the semidirect product of $\mathcal{F}^{-1}$ and $\mathcal{G}$, and the semidirect product of $\mathrm{GL}(\mathcal{F}, d)$ and $\mathcal{G}$.
(With $d = 1$ and $A = \mathbb{Z}$ both reduce to the standard Riordan group. The second gives some kind of group operation on a $d \times d$ matrix of high-dimensional infinite arrays. The first does not.)

Thanks!