53318—LMS JNL—O' FARRELL— 6pp. Eβ18

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LIP 1 RATIONAL APPROXIMATION

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ABSTRACT

Let X be a compact subset of the complex plane $\mathbb{C}$. We prove that every C^1 function is the limit in Lip $(1, X)$ norm of a sequence of rational functions if and only if X is a subset of a finite union of disjoint simple C^1 curves.

1. Let X be a compact subset of the complex plane $\mathbb{C}$. The space Lip $(1, X)$ consists of all those continuous complex-valued functions f on X for which the norm

$$\|f\|_{1,X} = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|} : x, y \in X, x \neq y\right\}$$

is finite. When given the norm

$$\|f\|_X = \|f\|_u + \|f\|_{1,X}$$

where $\|f\|_u$ denotes the supremum of $|f|$ on X, then Lip$(1, X)$ becomes a Banach algebra. This paper concerns the closure in Lip$(1, X)$ of the algebra $\mathscr{R}(X)$ of all rational functions with poles off X. It is obvious that the closure $R^1(X)$ of $\mathscr{R}$ is a subalgebra of the closure $D^1(X)$ of the space $C^1(\mathbb{C})$ of all continuously-differentiable functions on $\mathbb{C}$. Our main result gives a necessary and sufficient condition for $R^1(X)$ to equal $D^1(X)$.

THEOREM A. *$R^1(X) = D^1(X)$ if and only if X is a subset of a finite union of disjoint simple C^1 curves.*

By a simple C^1 curve we mean either a simple closed C^1 curve or a simple open C^1 curve (a simple closed C^1 curve is the image of a periodic C^1 function $\phi : \mathbb{R} \to \mathbb{C}$ which has non-vanishing derivative and is one-to-one inside each period; a simple open C^1 curve is the image the closed interval $[0, 1]$ under a one-to-one C^1 function $\phi : \mathbb{R} \to \mathbb{C}$ which has non-vanishing derivative).

The reader will find that the proof is much more accessible than is usual in rational approximation theory. All we use is some elementary function theory and a little geometry.

2. Let $a \in X$. We denote by Tan (X, a) the set of unit vectors u which are obtained as

$$u = \lim_{n \to +\infty} \frac{x_n - y_n}{|x_n - y_n|}$$

with $x_n \to a$, $y_n \to a$, $x_n \in X$, $y_n \in X$ (this set is larger, in general, than the " Tan (X, a) "

Received 21 May, 1974.

†Supported by NSF GP 33693X.

of [3]). We shall prove the theorem by showing that *the following three conditions are equivalent, for a compact set* $X \subset \mathbb{C}$:

(1) $R^1(X) = D^1(X)$;

(2) $\mathrm{Tan}\,(X, a)$ *has at most one-dimensional span* (*over* $\mathbb{R}$) *for each* $a \in X$;

(3) X *is a subset of a finite union of pairwise disjoint simple* C^1 *curves.*

To see that (1) implies (2), suppose (2) fails, so that there exist $a \in X$ and u, $v \in \mathrm{Tan}\,(X, a)$ with $u \neq \pm v$. By considering difference quotients it is easy to see that the directional derivatives

$$\langle u, Df(a)\rangle, \quad \langle v, Df(a)\rangle,$$

defined for $f \in C^1(\mathbb{C})$, extend to unique bounded point derivations on $D^1(X)$. Let us denote these derivations by the symbols D_u and D_v. Then the continuous linear functional L, defined by

$$Lf = vD_u f - uD_v f$$

whenever $f \in D^1(X)$, annihilates $\mathscr{R}$ but not $D^1(X)$. Hence (1) fails.

Before proving that (3) implies (1) we make two remarks. First, Runge's Theorem works in Lip (1, X) norm, i.e. for any compact $X \subset \mathbb{C}$, $\mathscr{R}(X)$ is dense in the space $\tilde{\mathscr{R}}(X)$ of all functions f in $C^1(\mathbb{C})$ which are analytic on some neighbourhood (depending on f) of X. In fact, the Riemann sums used in the usual proof of Runge's Theorem constitute a normal family on a neighbourhood of X, and hence converge in the C^1 topology on that neighbourhood. Second, if X is a C^1 curve, then $D^1(X)$ coincides with the usual manifold space $C^1(X)$, consisting of those continuous functions f on X whose derivative

$$\frac{df}{ds}$$

with respect to arc length is also continuous on X. Moreover the Lip (1, X) norm $\|f\|_u + \|f\|_1$ is comparable to the $C^1(X)$ norm

$$\|f\|_u + \left\|\frac{df}{ds}\right\|_u.$$

Now suppose X is a closed C^1 curve, and let $f \in C^1(X)$. Then

$$\frac{df}{dz}$$

is a continuous function on X, where

$$\frac{df}{dz} = \frac{df}{ds}\left(\frac{dz}{ds}\right)^{-1}$$

Thus (*cf.* [**1**; 4]) we may choose a sequence $r_n \in \mathscr{R}$ such that

$$\left\|r_n - \frac{df}{dz}\right\|_u < \frac{1}{n}$$

Each r_n may be assumed to have at most one pole, and since

$$\oint_X \frac{df}{dz}\,dz = 0,$$

each r_n may be assumed to have residue zero inside X. Hence there are functions $s_n \in \mathscr{R}$ such that

$$\frac{ds_n}{dz} = r_n.$$

We may assume that each s_n takes the same value as f at some point $a \in X$. We have

$$\left|\frac{ds_n}{ds} - \frac{df}{ds}\right| = \left|\frac{dz}{ds}\right| \left|\frac{ds_n}{dz} - \frac{df}{dz}\right| < \frac{1}{n}$$

and $|s_n - f| \leqslant l(X).1/n$, where $l(X)$ denotes the length of X. Hence the sequence $\{s_n\}$ converges to f in $C^1(X)$ norm, and hence in Lip $(1, X)$ norm. Thus $R^1(X) = D^1(X)$ whenever X is a simple closed C^1 curve.

Now suppose X is a union of a finite number of disjoint simple C^1 curves, say

$$X = X_1 \cup X_2 \cup \ldots \cup X_m.$$

Let χ_i denote the characteristic function of X_i. Each χ_i has an extension in $\tilde{\mathscr{R}}(X)$, and hence each χ_i is a limit of rational functions in Lip $(1, X)$ norm. Fix $f \in C^1(\mathbb{C})$ and $\varepsilon > 0$. Since each X_i is a subset of a closed C^1 curve, there exist rational functions $r_1, \ldots, r_2, \ldots, r_m$, with poles off X, such that

$$\|f - r_i\|_{X_i} < \varepsilon,$$

where $\|\cdot\|_{X_i} = \|\cdot\|_{u, X_i} + \|\cdot\|_{1, X_i}$ is the Lip $(1, X_i)$ norm. Also, there exist functions $s_i \in \mathscr{R}(X)$ such that

$$\|\chi_i - s_i\|_X < \varepsilon/M$$

where

$$M = 1 + \varepsilon + \|f\|_X + \sum_{i=1}^{m} \|r_i\|_X.$$

Let

$$d = \inf\{\text{dist}\,(X_i, X_j) : i \neq j\}.$$

Then

$$\|(\chi_i - s_i) f\|_X \leqslant \|\chi_i - s_i\|_X \, \|f\|_X < \varepsilon,$$

$$\|s_i(f - r_i)\|_{u, X} < (1 + \varepsilon)\varepsilon,$$

$$\|s_i(f - r_i)\|_{1, X} < (1 + \varepsilon)\,\varepsilon(1 + 2d^{-1}),$$

$$\|s_i(f - r_i)\|_X < (1 + \varepsilon)\,\varepsilon(2 + 2d^{-1}),$$

$$\left\|f - \sum_{i=1}^{m} r_i s_i\right\|_X \leqslant \sum_{i=1}^{m} \|(\chi_i - s_i) f\|_X + \sum_{i=1}^{m} \|s_i(f - r_i)\|_X$$

$$< m\varepsilon + m(1 + \varepsilon)\,\varepsilon(2 + 2d)^{-1}).$$

Thus $R^1(X) = D^1(X)$, and we have shown that (3) implies (1).

3. It remains to show that (2) implies (3). This is a purely geometric fact. It is equally valid in $\mathbb{R}^n$, as will be clear from the proof.

Suppose (2) holds. Given $a \in X$ and $u \in \mathrm{Tan}(X, a)$, there is a function $\varepsilon(r)$, which decreases to zero as r decreases to zero, such that

$$\left|\frac{x-y}{|x-y|} - u\right| \leqslant \varepsilon(r)$$

or (*)

$$\left|\frac{x-y}{|x-y|} + u\right| \leqslant \varepsilon(r)$$

holds whenever $x, y \in X$ with $|x-a| < r$, $|y-a| < r$. It follows that there exists a closed disc D centred at a such that the orthogonal projection of $D \cap X$ on the line

$$\{\alpha u : \alpha \in \mathbb{R}\}$$

is invertible, and that the inverse is in Lip 1 [3; (3.3.5)]. Thus there is a one-to-one Lip 1 map F defined on a subset Y of $\mathbb{R}$ with values in $\mathbb{C}$ such that

$$D \cap X = \mathrm{im}\, F.$$

F has a derivative $F'(t)$ at each accumulation point t of Y, in the sense that

$$\frac{1}{|u-t|}\, |F(u) - F(t) - F'(t)\,(u-t)| \to 0$$

as $|u-t| \downarrow 0$ with $u \in Y$ (in fact, the above expression is bounded by $\|F_1\|\, \varepsilon[\|F_1\|\, |u-t|]$). Also (*) implies that F' is continuous on its domain. By Whitney's Extension Theorem ([3; (3.1.14)], [7; Ch. VI]), there is a function $G \in C^{-1}(\mathbb{R})$ which agrees with F on Y. Since F inverts a projection, $|F'| \geqslant 1$ on dom F', hence G' does not vanish on a neighbourhood of dom F'. Thus G may be modified to produce a function $H \in C^{-1}(\mathbb{R})$ which agrees with F on Y, and has non-vanishing derivative. Thus there is a C^1 curve Γ such that $D \cap X = \Gamma \cap X$.

We have proved that to every point $a \in X$ there corresponds a closed disc $D(a)$ centred at a and a C^1 curve $\Gamma(a) \subset D(a)$ such that $X \cap \Gamma(a) = X \cap D(a)$. Clearly, we may also require that $\Gamma(a)$ meet bdy $D(a)$ in exactly two points, and that the angles of intersection exceed $\pi/4$. Now choose a finite collection of these discs covering X, say $D_1, D_2, \ldots, D_m$, and let the associated C^1 curves be $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$. Take $\Gamma_1{}^1 = \Gamma_1$. Modify Γ_2, if necessary, so that $\Gamma_2 \cap \mathrm{bdy}\, D_1$ contains at most two points, without disturbing the fact that $X \cap \Gamma_2 = X \cap D_2$. This is always possible since

$$\Gamma_2 \cap X \cap D_1 = \Gamma_1 \cap X \cap D_2,$$

and $\Gamma_1 \cap \mathrm{bdy}\, D_1$ contains two points. Then set $\Gamma_2{}^1 = \mathrm{clos}\,(\Gamma_2 \backslash D_1)$, so that $\Gamma_2{}^1$ is contained in a union of at most two disjoint C^1 arcs. Modify Γ_3 so that each of the sets $\Gamma_3 \cap \mathrm{bdy}\, D_1$ and $\Gamma_3 \cap \mathrm{bdy}\, D_2$ contains at most two points, and form $\Gamma_3{}^1 = \mathrm{clos}\,(\Gamma_3 \diagdown D_1 \diagdown D_2)$. Then $\Gamma_3{}^1$ is contained in a union of at most four disjoint C^1 arcs. Continuing in this way we obtain a collection $\Gamma_1{}^2, \Gamma_2{}^2, \ldots, \Gamma_M{}^2$ of C^1 arcs, with $M \leqslant 2^{m+1}$, such that $X \subset \bigcup_i \Gamma_i{}^2$, and $\Gamma_i{}^2 \cap \Gamma_j{}^2$ contains at most two points for $i \neq j$. If $a \in \Gamma_i{}^2 \cap \Gamma_j{}^2$, there are two possibilities.

Case 1. $\operatorname{Tan}(\Gamma_i^2 \cap X, a) \cap \operatorname{Tan}(\Gamma_j^2 \cap X, a) = \emptyset$. In this case one of the arcs may be shortened a little so that they no longer intersect at a, without disturbing the property that $X \subset \bigcup_i \Gamma_i^2$.

Case 2. $\operatorname{Tan}(\Gamma_i^2 \cap X, a) = \operatorname{Tan}(\Gamma_j^2 \cap X, a)$. In this case the two arcs may be joined together at a to form a new C^1 arc (they must approach a from opposite sides on account of the condition that the angle between Γ_i and bdy D_i exceeds $\pi/4$).

Thus, finally, we obtain a finite family $\{\Gamma_i^3\}$ of pairwise disjoint simple C^1 curves, such that $X \subset \bigcup_i \Gamma_i^3$, and so it is proved that (2) implies (3).

4. To illustrate the result, we give an example of a compact curve X which has a unique tangent direction at each point (in the classical sense) but is such that $R^1(X) \neq D^1(X)$. Let $f(x) = x^2 \sin x^{-2}$, and set

$$X = \{x+if(x) : 0 \leqslant x \leqslant 1\}.$$

Since $f'(x)$ exists for $0 \leqslant x \leqslant 1$, X has a tangent at each point. Also, since $f'(x)$ is continuous and unbounded on the interval $\{x : 0 < x < \eta\}$, no matter how small $\eta > 0$ may be, it follows that *every* $u \in \mathbb{C}$ with $|u| = 1$ belongs to $\operatorname{Tan}(X, 0)$. Thus condition (2) fails.

5. There is a higher order version of Theorem A. The space Lip $(n, \mathbb{C})$ consists of those bounded complex-valued functions on $\mathbb{C}$ which have bounded partial derivatives up to order $n-1$, and whose $(n-1)$th order derivatives belong to Lip 1. With the norm

$$\|f\|_n = \sum_{i+j \leqslant n-1} \left\| \frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right\|_u + \sum_{i+j=n-1} \left\| \frac{\partial^{n-1} f}{\partial x^i \partial y^j} \right\|_1,$$

Lip $(n, \mathbb{C})$ becomes a Banach algebra. For compact sets $X \subset \mathbb{C}$. the set

$$I(X) = \{f \in \operatorname{Lip}(n, \mathbb{C}) : f = 0 \text{ on } X\},$$

is a closed ideal in Lip $(n, \mathbb{C})$, and we define

$$\operatorname{Lip}(n, X) = \operatorname{Lip}(n, \mathbb{C})/I(X).$$

With the quotient norm, Lip (n, X) forms a Banach algebra. It may be regarded as a space of functions on X. There are more concrete descriptions of the space in terms of local properties of the functions [7]. We define $D^n(X)$ as the closure of $C^n(\mathbb{C})$ in Lip (n, X) and $R^n(X)$ as the closure of $\mathscr{R}(X)$.

THEOREM A′. *Let X be a compact subset of $\mathbb{C}$ and let n be a positive integer. Then $R^n(X) = D^n(X)$ if and only if X is contained in a finite disjoint union of simple C^n curves.*

The proof of this result is only slightly more complicated than that of Theorem A, and so we omit the details.

The approximation problem in fractional order Lipshitz spaces is radically different and will be treated elsewhere.

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