# CONTINUITY PROPERTIES OF HAUSDORFF CONTENT

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ABSTRACT

If $E$ is a subset of $\mathbb{R}^d$, contained in some ball with diameter 1, then the $\beta$-dimensional Hausdorff content $\mathbf{M}^\beta(E)$ is a non-decreasing function of $\beta$, for $\beta \geqslant 0$. For compact sets $E$, $\mathbf{M}^\beta(E)$ is continuous from the left. The main result of this paper is that if $0 \leqslant \alpha < \gamma \leqslant d$, and the lower $\gamma$-density of $E$ is uniformly bounded below at the points of $E$, then the right-hand limit of $\mathbf{M}^\beta(E)$ at $\alpha$ is at least as large as $\kappa \mathbf{M}^\gamma(E)$, where $\kappa$ is a positive number which depends only on $\alpha$, $\gamma$, and the lower bound for the density. The result has application to rational approximation in Lipschitz norms.

If $E$ is a subset of Euclidean space $\mathbb{R}^d$, and $\alpha$ is a non-negative real number, then the $\alpha$-dimensional Hausdorff content $\mathbf{M}^\alpha(E)$ is defined as the infimum of all sums

$$\sum_{S \in \mathscr{S}} (\operatorname{diam} S)^\alpha$$

where $\mathscr{S}$ runs over all countable coverings of $E$ by closed balls [**1**]. The content $\mathbf{M}^\alpha$ is also referred to as a size $\infty$ approximating Hausdorff measure [**2**; (2.10.1)]. Clearly $\mathbf{M}^\alpha(E)$ increases as $E$ increases and as $\alpha$ decreases (provided $E$ fits inside some ball with diameter 1). The continuity properties of $\mathbf{M}^\alpha(E)$ as a function of $E$ are well understood [**1**; Chapter 2], but the problem of how $\mathbf{M}^\alpha(E)$ behaves as $\alpha$ varies appears not to have been studied. The reason for this is, presumably, that most attention has been focused on the Hausdorff outer measure $\mathfrak{H}^\alpha(E)$ [**2**; (2.10.2)], and this is uninteresting as a function of $\alpha$. Indeed, $\mathfrak{H}^\alpha(E)$ is finite and non-zero for at most one value of $\alpha$. In contrast, $\mathbf{M}^\alpha(E)$ is finite for all $\alpha \geqslant 0$ whenever $E$ is bounded. The measure $\mathfrak{H}^\alpha$ and the content $\mathbf{M}^\alpha$ have the same null-sets, and so for some applications it is a matter of indifference which one is used. In the study of surface area, Green's theorem, and Plateau's problem [**2**], its properties as a Borel regular outer measure make $\mathfrak{H}^\alpha$ the preferred tool. But for approximation problems concerning analytic functions in the plane we find that $\mathbf{M}^\alpha$ is more suitable: the sharp quantitative conditions for approximation involve $\mathbf{M}^\alpha$ [**3, 4, 5**].

Before stating the theorem, we fix some notation. If $D$ is a closed (resp., open) ball, and $\tau$ is positive, then $\tau D$ denotes the concentric closed (resp., open) ball with radius equal to $\tau$ times the radius of $D$. If $E$ is a subset of $\mathbb{R}^d$, and $\beta$ is positive, then the *lower $\beta$-density* of $E$ at a point $x \in \mathbb{R}^d$ equals

$$\liminf_{\tau \downarrow 0} \frac{\mathbf{M}^\beta(E \cap \tau D)}{(\operatorname{diam} \tau D)^\beta}$$

where $D$ is any closed ball with centre $x$.

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THEOREM. *Let $E$ be a subset of $\mathbb{R}^d$, contained in some ball with diameter* 1.

(a) *If $E$ is closed, then*

$$\lim_{\beta \uparrow \alpha} \mathbf{M}^\beta(E) = \mathbf{M}^\alpha(E) \quad (0 < \alpha < \infty)$$

(b) *If $\Gamma > 0 \leqslant \alpha < \gamma \leqslant d$, and the lower $\gamma$-density of $E$ at each point of $E$ is no less than $\Gamma$, then*

$$\kappa \mathbf{M}^\alpha(E) \leqslant \lim_{\beta \downarrow \alpha} \mathbf{M}^\beta(E) \leqslant \mathbf{M}^\alpha(E),$$

*where*

$$\kappa = \{2 + \Gamma^{-1/\gamma}\}^{-\alpha\gamma/(\gamma-\alpha)}.$$

Part (a) is quite simple, as we shall see. We will give examples to show that (a) fails for some open sets and (b) fails for some closed sets. We will give an application of (b) to rational approximation in Lipschitz norms.

We do not know whether or not the constant $\kappa$ may be replaced by 1.

## 1. *Proof of Theorem*

(a) Clearly

$$\lim_{\beta \uparrow \alpha} \mathbf{M}^\beta(E) \geqslant \mathbf{M}^\alpha(E).$$

Fix $\varepsilon > 0$, and choose a countable covering $S$ of $E$ by closed balls with

$$\sum_{S \in \mathscr{S}} (\text{diam } S)^\alpha < \mathbf{M}^\alpha(E) + \varepsilon.$$

Replacing each ball $S$ in $\mathscr{S}$ by a slightly larger open ball, we obtain a covering $\mathscr{T}$ of $E$ such that

$$\sum_{T \in \mathscr{T}} (\text{diam } T)^\alpha < \mathbf{M}^\alpha(E) + \varepsilon.$$

Let $\mathscr{T}_1$ be a finite subcover selected from $\mathscr{T}$. For each $T \in \mathscr{T}_1$,

$$\lim_{\beta \uparrow \alpha} (\text{diam } T)^\beta = (\text{diam } T)^\alpha;$$

hence there exists a number $\gamma < \alpha$ such that

$$\mathbf{M}^\beta(E) \leqslant \sum_{T \in \mathscr{T}_1} (\text{diam } T)^\beta < \mathbf{M}^\alpha(E) + \varepsilon$$

whenever $\gamma < \beta < \alpha$; hence

$$\lim_{\beta \uparrow \alpha} \mathbf{M}^\beta(E) \leqslant \mathbf{M}^\alpha(E) + \varepsilon.$$

The result follows.

(b) The right-hand inequality is obvious. We divide the proof of the left-hand inequality into steps.

*Step* 1. Fix $\mu$, with

$$0 < \mu < \{2 + (4/\Gamma)^{1/\gamma}\}^{-\gamma/(\gamma-\alpha)} < \tfrac{1}{2}.$$

For $\beta > \alpha$, let us say that a family $\mathscr{T}$ of closed balls is *$\beta$-vulnerable* with respect to a closed ball $D$ if $\bigcup \mathscr{T} \subset D$ and

$$(\operatorname{diam} D)^\beta < (1-\mu) \sum_{T \in \mathscr{T}} (\operatorname{diam} T)^\beta.$$

When no confusion is possible we will write " vulnerable " instead of " $\beta$-vulnerable ".

For each $\beta$, with $\alpha < \beta < d$, we claim that *there exists a countable covering $\mathscr{S}_\beta$ of $E$ by closed balls with the following properties*:

(1) $\sum_{S \in \mathscr{S}_\beta} (\operatorname{diam} S)^\beta < (1+\beta-\alpha)\, \mathbf{M}^\beta(E)$,

(2) *no subfamily of $\mathscr{S}_\beta$ is $\beta$-vulnerable, with respect to any ball.*

Let $\rho$ denote the Hausdorff metric [2, 3] on the closed subsets of $\mathbb{R}^d$. Recall that, with respect to the topology induced by $\rho$, bounded sets have compact closure.

Choose a countable covering

$$\mathscr{S}_0 = \{S_1, S_2, S_3, \ldots\}$$

of $E$ by closed balls such that each $S_k$ meets $E$, the diameters of the $S_k$ are non-increasing, and $\mathscr{S}_0$ satisfies (1) (with $\mathscr{S}_\beta$ replaced by $\mathscr{S}_0$). If $\mathscr{S}_0$ satisfies (2), the proof is complete. Otherwise, let $\mathscr{B}_0$ denote the family of all closed balls $D$ such that some subfamily of $\mathscr{S}_0$ is vulnerable with respect to $D$. Let

$$m_1 = \sup \{\operatorname{diam} D : D \in \mathscr{B}_0\}.$$

Choose a sequence $\{B_i\}_{i=1}^\infty$ contained in $\mathscr{B}_0$ such that diam $B_i \uparrow m_i$ as $i \uparrow \infty$. If $B \in \mathscr{B}_0$, then $B$ meets $E$, and

$$\operatorname{diam} \mathrm{B} \leqslant \{(1-\mu)(1+\beta-\alpha)\, \mathbf{M}^\beta(E)\}^{1/\beta}.$$

Hence, letting

$$\sigma = \operatorname{diam} E + \{(1-\mu)(1+\beta-\alpha)\, \mathbf{M}^\beta(E)\}^{1/\beta},$$

we have

$$\mathscr{B}_0 \subset \{A \subset \mathbb{R}^d : A \text{ is closed, } \rho(A, \operatorname{clos} E) \leqslant \sigma\}$$

so that $\mathscr{B}_0$ has compact closure. Thus there exists a subsequence of $\{B_i\}_i$ converging to a closed ball $B_\infty$ with diameter $m_1$. Let

$$D_1 = (1-\mu)^{-1/(2\beta)} B_\infty.$$

Then $D_1$ contains a ball $B \in \mathscr{B}_0$ with

$$\operatorname{diam} B > (1-\mu)^{1/(2\beta)} m_1 = (1-\mu)^{1/\beta} \operatorname{diam} D_1.$$

Let $\mathscr{T}_1$ be a subfamily of $\mathscr{S}_0$ that is vulnerable with respect to $B$. Then $\bigcup \mathscr{T}_1 \subset D_1$, and

$$(\operatorname{diam} D_1)^\beta < \sum_{S \in \mathscr{T}_1} (\operatorname{diam} S)^\beta.$$

Let $\mathscr{S}_1$ denote the family

$$\{D_1\} \cup \mathscr{S}_0 \sim \mathscr{T}_1.$$

Then $\mathscr{S}_1$ covers $E$, since $D_1 \notin \mathscr{T}_1$. Clearly,

$$\sum_{S \in \mathscr{S}_1} (\operatorname{diam} S)^\beta < \sum_{S \in \mathscr{S}_0} (\operatorname{diam} S)^\beta,$$

so that $\mathscr{S}_1$ satisfies (1). Enumerate $\mathscr{S}_1$ as

$$S_1{}^1, S_2{}^1, S_3{}^1, \ldots,$$

with diam $S_k^1$ non-increasing. If several elements have the same diameter as $D_1$ place $D_1$ last.

If $\mathscr{S}_1$ contains no vulnerable subfamilies, the proof is complete. Otherwise, let $\mathscr{B}_1$ denote the family of all closed balls $D$ such that some subfamily of $\mathscr{S}_1$ is vulnerable with respect to $D$, and form $m_2$, $D_2$, $\mathscr{T}_2$ and $\mathscr{S}_2$ in the same way as before. If $\mathscr{S}_2$ contains no vulnerable subfamily, stop. Otherwise, continue. If the process stops at the $p$-th stage, then $\mathscr{S}_p$ is the desired covering.

Suppose the process does not stop. Let $p$ be a positive integer and let $\mathscr{T}$ be a subfamily of $\mathscr{S}_p$ that is vulnerable with respect to some closed disc $D$. Write $\mathscr{T} = \mathscr{G} \cup \mathscr{H}$, where $\mathscr{G} \subset \mathscr{S}_0$ and $\mathscr{H} \subset \{D_1, D_2, \ldots, D_p\}$. We claim that $\mathscr{H} = \varnothing$. For suppose not, and let $\mathscr{K}$ be the union of the $\mathscr{T}_j$ corresponding to the various $D_j$ in $\mathscr{H}$. Let $q$ be the least index such that $D_q \in \mathscr{H}$. Then, since $\mathscr{T}_i \cap \mathscr{T}_j = \varnothing$ $(i \neq j)$ and $\mathscr{K} \cap \mathscr{G} = \varnothing$, we have

$$(\text{diam } D)^\beta < (1-\mu) \sum_{T \in \mathscr{G} \cup \mathscr{H}} (\text{diam } T)^\beta$$

$$< (1-\mu) \sum_{T \in \mathscr{G} \cup \mathscr{K}} (\text{diam } T)^\beta,$$

hence $D \in \mathscr{B}_{q-1}$. Since diam $D >$ diam $D_q > m_q$, we have a contradiction. Thus $\mathscr{H} = \varnothing$, hence every vulnerable subfamily of $\mathscr{S}_p$ is a subfamily of $\mathscr{S}_0$. This has two consequences. First, since

$$\mathscr{S}_0 \supset \mathscr{S}_1 \cap \mathscr{S}_0 \supset \mathscr{S}_2 \cap \mathscr{S}_0 \supset \ldots,$$

it follows that

$$\mathscr{B}_0 \supset \mathscr{B}_1 \supset \mathscr{B}_2 \supset \ldots,$$

hence $m_j \downarrow$, and so diam $D_j \downarrow$, as $j \uparrow$. Second, no $D_j$ is ever removed in the process, so $\mathscr{S}_p$ contains $D_1, D_2, \ldots, D_p$, and in that order.

Since each $\mathscr{S}_p$ satisfies (1), diam $D_j \downarrow 0$ as $j \uparrow \infty$. Let $S \in \mathscr{S}_0$. Then there is an integer $t$ such that for all $j \geqslant t$, diam $D_j <$ diam $S$, hence $m_j <$ diam $S$; hence $S$ does not belong to any vulnerable subfamily of $\mathscr{S}_{j-i}$. Thus if $S$ is not removed by the $t$-th stage, then it is not removed at all, and moreover its place in $\mathscr{S}_j$ has the same index for all $j \geqslant t$. It follows that each $D_q$ also has the same index in $\mathscr{S}_j$ for all large $j$. Let $\mathscr{S}_\infty$ denote the family $\{S_1^\infty, S_2^\infty, S_3^\infty, \ldots\}$, defined by setting $S_p^\infty$ equal to the ultimate value of $S_p^j$ for large $j$. Each $S \in \mathscr{S}_0$ either belongs to $\mathscr{S}_\infty$ or is contained in some $D_j \in \mathscr{S}_\infty$, hence $\bigcup \mathscr{S}_0 \subset \bigcup \mathscr{S}_\infty$, and $\mathscr{S}_\infty$ covers $E$. It is easy to see that $\mathscr{S}_\infty$ satisfies (1). Suppose $\mathscr{S}^\infty$ contains a subfamily $\mathscr{T}$ vulnerable with respect to some disc $D$. Decomposing $\mathscr{T}$ as before into $D_j$'s and $S_j$'s, we deduce that $\mathscr{T}$ consists entirely of $S_j$'s. Since no $S_j$ in $S_\infty$ is contained in a vulnerable subfamily, we have a contradiction. Hence $S_\infty$ satisfies (2), and the claim is proved.

*Step* (II). We may assume that each element of each family $\mathscr{S}_\beta$ meets $E$. This means that the family $\bigcup \{\mathscr{S}_\beta : \beta > \alpha\}$ is a subfamily of the compact family

$$\{A \subset \mathbb{R}^d : A \text{ is closed}, \rho(A, \text{clos } E) \leqslant 3\},$$

and hence has compact closure. Select a sequence $\beta_j \downarrow \alpha$ with $\beta_1 < \gamma$, let $\mathscr{R}^j$ denote $\mathscr{S}_{\beta_j}$, and let

$$\mathscr{R}^j = \{R_1^j, R_2^j, R_3^j, \ldots\}, d_k^j = \text{diam } R_k^j.$$

Assume that $d_1^j \geqslant d_2^j \geqslant d_3^j \geqslant \ldots$, for each $j$. Let $\mathscr{D}_1$ denote the family of all closed balls $D \subset \mathbb{R}^d$ such that

$$\mathbf{M}^\gamma(E \cap \tau D) \geqslant \Gamma(1-\mu)(\operatorname{diam} \tau D)^\gamma$$

whenever $0 < \tau \leqslant 1$. Let

$$\delta_1 = \sup\{\operatorname{diam} D : D \in \mathscr{D}_1\}.$$

There is a positive integer $N$ such that

$$\{2+\Gamma^{-1/\gamma}(1-\mu)^{-2/\gamma}\}^{-\gamma/(\gamma-\beta_j)} > \{2+\Gamma^{-1/\gamma}(1-\mu)^{-2/\gamma}\}^{-\gamma/(\gamma-\alpha)} - \mu,$$

whenever $j \geqslant N$. Let $\eta$ denote the right-hand side in this inequality. Observe that $0 < \eta < 1/2$.

*Claim.* $d_1{}^j > \eta\delta_1$ *whenever* $j \geqslant N$.

*Proof of Claim.* Suppose the claim fails, so that

$$d_k{}^j \leqslant \eta\delta_1$$

for some $j \geqslant N$ and every $k$. We seek a contradiction.

Let $\lambda$ be given, $0 < \lambda < 1-2\eta$. Choose a closed ball $D$ belonging to $\mathscr{D}_1$, with diameter $(1-\lambda)\delta_1$. If $R \in \mathscr{R}^j$ and $R \sim D$ is non-empty, then $R$ does not meet the ball

$$D_1 = \left(\frac{1-\lambda-2\eta}{1-\lambda}\right) D.$$

Let $\mathscr{T}$ denote the family of all balls in $\mathscr{R}^j$ which meet $D_1$. Then $\mathscr{T}$ covers $E \cap D_1$, and $\bigcup \mathscr{T} \subset D$. Hence, with $\beta = \beta_j$, we have

$$(1-\lambda-2\eta)^\gamma \delta_1{}^\gamma \leqslant \Gamma^{-1}(1-\mu)^{-1} \sum_{S \in \mathscr{T}} (\operatorname{diam} S)^\gamma$$

$$\leqslant \Gamma^{-1}(1-\mu)^{-1}(\eta\delta_1)^{\gamma-\beta} \sum_{S \in \mathscr{T}} (\operatorname{diam} S)^\beta;$$

hence

$$\sum_{S \in \mathscr{T}} (\operatorname{diam} S)^\beta \geqslant \frac{\Gamma(1-\mu)(1-\lambda-2\eta)^\gamma \delta_1{}^\beta}{\eta^{\gamma-\beta}}.$$

Since $\mathscr{T}$ is invulnerable,

$$\Gamma(1-\mu)^2(1-\lambda-2\eta)^\gamma \leqslant (1-\lambda)^\beta \eta^{\gamma-\beta}.$$

Since this holds for all small positive $\lambda$, we deduce that

$$\Gamma(1-\mu)^2(1-2\eta)^\gamma \leqslant \eta^{\gamma-\beta},$$

$$1 \leqslant 2\eta + \Gamma^{-1/\gamma}(1-\mu)^{-2/\gamma}\eta^{(\gamma-\beta)/\gamma} < \{2+\Gamma^{-1/\gamma}(1-\mu)^{-2/\gamma}\}\eta^{(\gamma-\beta)/\gamma} \leqslant 1.$$

This is the desired contradiction.

*Step* (III). Hence the sequence $\{d_1{}^j\}_{j=N}^\infty$ is bounded below by $\eta\delta_1$, and hence $\{R_1{}^j\}_{j=1}^\infty$ contains a subsequence $\{R_1{}^j : j \in J_1\}$ which converges to a closed ball $R_1$ such that

$$\operatorname{diam} R_1 \geqslant \eta\delta_1, \; R_1 \cap \operatorname{clos} E \neq \varnothing.$$

If $E \subset (1+\mu) R_1$, we stop. Otherwise, for large $j \in J_1$, the family $\mathscr{R}^j \sim \{R_1{}^j\}$ covers the set $E \sim (1+\mu) R_1$. Defining $\mathscr{D}_2$ in the same way as $\mathscr{D}_1$, but with $E$ replaced by $E \sim (1+\mu) R_1$, and setting

$$\delta_2 = \sup \{\operatorname{diam} D : D \in \mathscr{D}_2\},$$

we have $d_2{}^j > \eta\delta_2$ for large $j \in J_1$, as in Step (II). Thus the sequence $\{R_2{}^j : j \in J_1\}$ contains a subsequence $\{R_2{}^j : j \in J_2\}$ which converges to a closed ball $R_2$ with

$$\operatorname{diam} R_2 \geqslant \eta\delta_2, R_2 \cap \operatorname{clos} E \neq \varnothing.$$

If $E \subset (1+\mu) R_1 \cup (1+\mu) R_2$, we stop. Otherwise we continue.

If the process stops at the $p$-th stage we have a subsequence $\{\mathscr{R}^j : j \in J_p\}$ of $\{\mathscr{R}^j\}_1^\infty$ and a collection $\{R_1, R_2, \ldots, R_p\}$ of closed balls such that

$$\lim_{J_p \ni j \uparrow \infty} \rho(R_k{}^j, R_k) = 0 \qquad (1 \leqslant k \leqslant p),$$

$$E \subset \bigcup_{k=1}^{p} (1+\mu) R_k.$$

Thus, for $j \in J_p$ we may write

$$\begin{aligned}(1+\beta_j-\alpha)\, \mathbf{M}_j^\beta(E) &> \sum_{k=1}^{\infty} (\operatorname{diam} R_k{}^j)^{\beta_j}\\ &\geqslant \sum_{k=1}^{p} (\operatorname{diam} R_k{}^j)^{\beta_j}\\ &\to \sum_{k=1}^{p} (\operatorname{diam} R_k)^\alpha\\ &\geqslant (1+\mu)^{-1}\, \mathbf{M}^\alpha(E);\end{aligned}$$

hence

$$\mathbf{M}^\alpha(E) \leqslant (1+\mu) \lim_{\beta \downarrow \alpha} \mathbf{M}^\beta(E).$$

If the process does not stop, then by diagonalising we obtain a subsequence $\{\mathscr{R}^j : j \in J\}$ of $\{\mathscr{R}^j\}_1^\infty$, and a sequence $\{R_1, R_2, R_3, \ldots\}$ of closed balls such that

$$\lim_{J \ni j \uparrow \infty} \rho(R_k{}^j, R_k) = 0 \qquad (k = 1, 2, 3, \ldots).$$

Let $x \in E$, and let

$$\delta = \sup \{\operatorname{diam} D : D \in \mathscr{D}_1, x = \text{centre of } D\}.$$

Then, by the argument of Step (II), each covering $\mathscr{R}^j (j \geqslant N)$ has a ball of diameter at least $\eta\delta$ which meets the closed ball $B$ with centre $x$ and diameter $(1-2\eta)\,\delta$. Thus there exists an integer $k$ such that $R_k$ meets $B$ and $\operatorname{diam} R_k \geqslant \eta\delta$, and hence $x \in \eta^{-1} R_k$. Thus the family $\{\eta^{-1} R_k\}_{k=1}^\infty$ covers $E$, and hence

$$\begin{aligned}\mathbf{M}^\alpha(E) &\leqslant \eta^{-\alpha} \sum_{k=1}^{\infty} (\operatorname{diam} R_k)^\alpha\\ &= \eta^{-\alpha} \sum_{k=1}^{\infty} \lim_{J \ni j \uparrow \infty} (\operatorname{diam} R_k{}^j)^{\beta_j}\end{aligned}$$

$$\leqslant \eta^{-\alpha} \liminf_{J \ni j \uparrow \infty} \sum_{k=1}^{\infty} (\operatorname{diam} R_k^{\,j})^{\beta_j}$$

$$< \eta^{-\alpha} \lim_{J \ni j \uparrow \infty} (1+\beta_j-\alpha)\, \mathbf{M}^{\beta_j}(E)$$

$$= \eta^{-\alpha} \lim_{\beta \downarrow \alpha} \mathbf{M}^{\beta}(E).$$

We used Fatou's lemma to obtain the third line.

Thus, whether the process stops or not,

$$\mathbf{M}^{\alpha}(E) \leqslant \eta^{-\alpha} \lim_{\beta \downarrow \alpha} \mathbf{M}^{\beta}(E).$$

Letting $\mu$ tend to zero, we get

$$\mathbf{M}^{\alpha}(E) \leqslant \{2+\Gamma^{-1/\gamma}\}^{\alpha\gamma/(\gamma-\alpha)} \lim_{\beta \downarrow \alpha} \mathbf{M}^{\beta}(E),$$

which is the desired result. This concludes the proof of (b).

With regard to the question of replacing $\kappa$ by 1 in (b), we remark that this could be done if it could be shown that

$$\mathbf{M}^{\alpha}\left(E \sim \bigcup_{k=1}^{\infty} R_k\right)$$

tends to zero with $\mu$.

If we are willing to settle for the constant $\kappa 3^{-\alpha}$ instead of $\kappa$, then the proof of (b) can be simplified somewhat. Here is an outline of the simplified proof:

Fix $\mu$,

$$0 < \mu < \{2+(4/\Gamma)^{1/\gamma}\}^{-\gamma/(\gamma-\alpha)}.$$

Choose a countable family $\mathscr{V}$ of pairwise disjoint closed balls such that $\mathscr{V} \subset \mathscr{D}_1$ and such that, given $x \in E$ with $\delta = \sup\{\operatorname{diam} D : D \in \mathscr{D}_1, x = \text{centre } D\}$, there exists some ball $V \in \mathscr{V}$ with radius at least $(1-\mu)\delta$ such that $V$ meets the closed ball with centre $x$ and radius $\delta$. Combining a simpler version of the argument of Step (I) of the proof with the idea of Step (II), construct for each $\beta > \alpha$ a covering $\mathscr{S}_\beta$ of $E$ such that for each $V \in \mathscr{V}$, the ball $(1-2\eta)\,V$ meets some $S \in \mathscr{S}_\beta$ with $\operatorname{diam} S \geqslant \eta \operatorname{diam} V$. Then argue as in Step (II). The only difference is that if the process does not stop, then we get that $\{3(1-\mu)^{-1}\eta^{-1} R_k\}_{k=1}^{\infty}$ covers $E$ (instead of $\{\eta^{-1} R_k\}_1^{\infty}$).

## 2. *Examples and Application*

*Example (a).* Let $P$ denote the cube:

$$\{x \in \mathbb{R}^d : 0 \leqslant x_i \leqslant d^{-1/2} \quad \text{for each } i\}.$$

Let $0 < \alpha \leqslant d$. For each positive integer $m$ such that $1/m < \alpha$, there exists a totally disconnected compact set $E_m \subset P$ such that

$$\mathbf{M}^{\alpha}(E_m) = 0, \qquad \mathbf{M}^{\alpha-1/m}(E_m) = d^{-1/2}$$

[**2**; (2.10)]. Let $E$ be the union of the $E_m$. Then $E$ is sigma-compact, is contained in a ball of diameter 1, and

$$\mathbf{M}^{\alpha}(E) \leqslant \sum_{m} \mathbf{M}^{\alpha}(E_m) = 0 < d^{-1/2} = \lim_{\beta \uparrow \alpha} \mathbf{M}^{\beta}(E).$$

If $\varepsilon > 0$ is pre-assigned, we may fatten up each $E_m$ a little and get an open set $G$ such that

$$\mathbf{M}^{\alpha}(G) < \varepsilon \lim_{\beta \uparrow \alpha} \mathbf{M}^{\beta}(G).$$

Thus the conclusion of part (a) of the theorem fails for some open sets.

*Example (b).* Let $0 \leqslant \alpha < d$. Then there is a product Cantor set $E$ [**2**; (2.10)] such that

$$\mathbf{M}^{\alpha}(E) = 1, \quad \mathbf{M}^{\beta}(E) = 0 \quad (\beta > \alpha).$$

The left-hand inequality in (b) fails for this set, no matter what constant $\kappa$ is used.

*Application. A measure function* $h(r)$ is a non-negative increasing function defined for $r \geqslant 0$. To each measure function $h$ we associate the content $\mathbf{M}_h$ on $\mathbb{R}^d$, where $\mathbf{M}_h(E)$ is defined as the infimum of the sums

$$\sum_{S \in \mathscr{S}} h(\operatorname{diam} S),$$

where $\mathscr{S}$ runs over all countable coverings of $E$ by closed balls. Obviously $\mathbf{M}_h = \mathbf{M}^{\beta}$ when $h(r) = r^{\beta}$. For $\beta \geqslant 0$, the content $\mathbf{M}_*{}^{\beta}$ is defined by $\mathbf{M}_*{}^{\beta}(E) = \sup\{\mathbf{M}_h(E) : h$ is a measure function, $h(r) \leqslant r^{\beta}$, $r^{-\beta} h(r) \to 0$ as $r \downarrow 0\}$.

Let $X$ be a compact subset of the complex plane, $\mathbb{C}$, and let $0 < \alpha < 1$. The theorem of [**5**] gives a condition on $X$ that is necessary and sufficient in order that every function in lip $(\alpha, X)$ that is analytic on int $(X)$ be the limit in the Lip $(\alpha, X)$ norm of a sequence of rational functions. The condition is that there exist a constant $\mu > 0$ such that

$$\mathbf{M}^{1+\alpha}(D \sim X) \geqslant \mu \mathbf{M}_*{}^{1+\alpha}(D \sim \operatorname{int} X)$$

whenever $D$ is an open disc with diameter less than 1/2.

*Let* $1+\alpha < \gamma \leqslant 2$, *and assume that the lower* $\gamma$-*density of* $\mathbb{C} \sim \operatorname{int} X$ *is uniformly bounded below on* $\mathbb{C} \sim \operatorname{int} X$. *Then the above condition is equivalent to the existence of a constant* $\mu' > 0$ *such that*

$$\mathbf{M}^{1+\alpha}(D \sim X) \geqslant \mu' \mathbf{M}^{1+\alpha}(D \sim \operatorname{int} X),$$

*whenever D is an open disc with diameter less than* 1/2.

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