J. f. d. Reine u. Angew. Math.
311/312 (1979) 101–115

Estimates for capacities, and approximation in Lipschitz norms

By *Anthony G. O'Farrell* at Maynooth

§ 1. Introduction

(1. 1) This paper is about the qualitative theory of approximation in Lipα norm, for $0<\alpha<1$, by analytic functions of one complex variable. Our purpose is to isolate, and where possible resolve, the main problems remaining in this area. The reader should be familiar with our paper [14], in which we solved the first basic problem, namely: for which compact sets $X \subset \mathbb{C}$ is it true that every function in lipα which is analytic on the interior of X can be approximated by rationals in Lip(α, X) norm?

The problems we consider are suggested by the results and problems on L^p approximation [1], [9], [10], [11], [16] and uniform approximation [3], [5], [6], [7], [8], [12], [17], [19], [20].

(1. 2) First, what is the Lipα analogue of Vitushkin's "individual function theorem" for uniform rational approximation ([19], Lemma 1, p. 172, [6], viii. 8. 1, p. 214)? That is, when is a given lipα function approximable by rational functions with poles off a compact set X? This is known to be a local question [13] (as indeed are all questions of Lipα approximation involving the so-called T-invariant algebras [3]).

The problem is to give a local condition on a function f which is necessary and sufficient for it to be approximable by rationals. We give the answer in Theorem 1 (§ 3), in terms of Hausdorff content and the weak $\bar{\partial}$ derivative of f. The work of Davie [3], (2. 1), p. 412 suggests the possibility of finding a local characterization of the functions in the closure of an arbitrary T-invariant subalgebra of Lipα, but we have been unable to do this. We can only do it when the capacity associated with the T-invariant algebra is some kind of Hausdorff content.

(1. 3) Second, which are the "analytically negligible" sets for Lipα approximation (cf. [6], p. 234)? They turn out to be the same as the sets of removable singularities for lipα analytic functions, and are characterized metrically as the sets of lower $(1+\alpha)$-dimensional Hausdorff content zero (cf. Theorem 2, § 6). The analogous problem for uniform approximation is still unsolved.

0075-4102/79/0311-0006$02.00

(1. 4) Let $0<\alpha<\beta<1$, and let X be a compact subset of $\mathbb{C}$. When is every $\mathrm{Lip}\,\beta$ function which is analytic on the interior of X a limit in $\mathrm{Lip}(\alpha, X)$ norm of rational functions with poles off X? The analogous question for uniform approximation was answered by Mergelyan [8], (11. 3), and in a more concrete way by Melnikov [12], Theorem 3. The $\mathrm{Lip}\,\alpha$ solution is Theorem 3 (§ 8).

When is every $\mathrm{lip}\,\alpha$ function which is analytic on the interior of X a limit in $\mathrm{Lip}\,\alpha$ norm of $\mathrm{Lip}\,\beta$ functions analytic on the interior of X? We give a sufficient condition in Theorem 4 (§ 8). The problem remains open. The corresponding problem for uniform approximation also lacks a satisfactory solution. It has been solved in terms of capacities [3], (2. 3), p. 414, but the capacities have not been classified in metric or topological terms.

Two of the estimates obtained on the way to these results (Lemma (7. 6) and Lemma (7. 8)) are of independent interest. They relate the $\mathrm{Lip}\,\alpha$, $\mathrm{Lip}\,\beta$, and uniform norms of a function analytic off a compact set E, and involve Hausdorff contents of E.

(1. 5) Let E be a subset of the boundary of X. When is every $\mathrm{lip}\,\alpha$ function which is analytic on the interior of X a limit of $\mathrm{lip}\,\alpha$ functions analytic on the interior of X, and on a neighbourhood of E? This is related to problem (1. 3) above, and the techniques used to prove Theorem 2 give a solution, Theorem 5 (§ 9). The uniform analogue is [6], (viii. 7. 4), p. 213.

This problem, and those of (1. 4), are special cases of the problem: when do two T-invariant subalgebras of $\mathrm{lip}\,\alpha$ have the same closure? We gave a necessary condition in [14], Lemma 2, p. 189. We conjecture that this condition is sufficient [3], (2. 3), p. 414.

(1. 6) Apart from these approximation problems, there are many related problems. Among the most interesting we mention the following:

Let T be a continuous linear functional on $\mathrm{Lip}\,\alpha$ with compact support. Give a condition on the Cauchy transform $\hat{T}$ which is necessary and sufficient for T to annihilate the $\mathrm{Lip}\,\alpha$ functions analytic on an open set U [1]. If T is a *measure* (that is, continuous with respect to the uniform norm), then a necessary and sufficient condition is that $\hat{T}$ vanish $M^{1+\alpha}$ a.e. off U.

Which functions are Cauchy transforms of continuous linear functionals on $\mathrm{Lip}\,\alpha$ [7], (3. 5), p. 52?

Is the content $M^{\beta}(E)$ continuous from the left as a function of β for all open sets E [15]?

(1. 7) The results we obtain are generally more satisfying than those obtained in the uniform case, since they are phrased in terms of contents, and are thus more explicit. In this respect they match the L^p results, which are given in terms of classical capacities. Of course, it is still possible that the various uniform norm capacities may turn out to be comparable to integral-geometric contents.

§ 2. Notation

(2. 1) A *modulus* is a positive increasing function $\omega(r)$, defined for $r>0$, with $\omega(0+)=0$, such that $r/\omega(r)$ is also increasing.

Such a function is subadditive, hence is uniformly continuous and is its own modulus of continuity.

Let $\omega(r)$ be a modulus, and let E be a subset of the complex plane, $\mathbb{C}$. For a function $f: E \to \mathbb{C}$ and a number $r>0$, let $\omega_f(r)$ denote the modulus of continuity,

$$\omega_f(r) = \sup\{|f(x)-f(y)| : |x-y| \leqq r\},$$

and let

$$\|f\|^{\omega} = \sup\{\omega_f(r)/\omega(r) : r>0\}.$$

(this quantity may be $+\infty$). Let $0<\alpha<1$, and let $\omega(r)=r^{\alpha}$. Then $\omega(r)$ is a modulus, and we define

$$\|f\|_{\alpha,E} = \|f\|^{\omega}, \ \|f\|_{\alpha} = \|f\|_{\alpha,\mathbb{C}},$$

$$\mathrm{Lip}(\alpha, E) = \{f: \|f\|_{\alpha,E} < \infty\},$$

$$\mathrm{lip}(\alpha, E) = \{f \in \mathrm{Lip}(\alpha, E) : \omega_f(r)/r^{\alpha} \to 0 \quad \text{as} \quad r \downarrow 0\},$$

$$\mathrm{Lip}\,\alpha = \mathrm{Lip}(\alpha, \mathbb{C}), \ \mathrm{lip}\,\alpha = \mathrm{lip}(\alpha, \mathbb{C}).$$

We shall be concerned with approximation in $\|\cdot\|_{\alpha}$ seminorm. We remark that if $\|f_n - f\|_{\alpha} \to 0$, then there are constants α_n such that $f_n + \alpha_n \to f$ uniformly on compacta. Indeed, we may take $\alpha_n = f(0) - f_n(0)$.

(2. 2) For $f: \mathbb{C} \to \mathbb{C}^n$ we denote

$$\|f\|_{\infty} = \sup\{|f_1|^2 + \cdots + |f_n|^2\}^{\frac{1}{2}},$$

where $f=(f_1, \ldots, f_n\}$.

Let $\mathscr{D}$ denote the space of infinitely-differentiable functions $\phi: \mathbb{C} \to \mathbb{C}$ with compact support. For $\phi \in \mathscr{D}$, let $D(\phi)$ denote the smallest closed disc containing $\mathrm{spt}\,\phi$, and let $d(\phi) = \mathrm{diam}\, D(\phi)$,

$$N(\phi) = \|\phi\|_{\infty} + d(\phi)\,\|\nabla\phi\|_{\infty} \quad \text{where} \quad \nabla\phi = (\phi_x, \phi_y).$$

For $\phi \in \mathscr{D}$, the *Vitushkin localization operator* T_{ϕ} is the linear operator defined by

$$(T_{\phi} f)(z) = f(z)\,\phi(z) - \frac{1}{\pi}\int \frac{f(\zeta)\,\bar{\partial}\phi(\zeta)}{\zeta - z}\, d\mathscr{L}^2(\zeta)$$

for all functions $f: \mathbb{C} \to \mathbb{C}$ and all $z \in \mathbb{C}$ for which the right-hand side exists. Here $\mathscr{L}^2$ is two-dimensional Lebesgue measure, and

$$\bar{\partial}\phi = \tfrac{1}{2}(\phi_x + i\phi_y).$$

If f is locally L^p, for any $p>2$, then $T_{\phi} f$ exists everywhere, and is analytic wherever f is analytic, and off $\mathrm{spt}\,\phi$. Also, $f - T_{\phi} f$ is analytic on the interior of the set $\phi^{-1}(1)$ (cf. [6], [7], [19], [20]).

(2. 3) Let $h(r)$ be positive and increasing for positive r. For any set $E \subset \mathbb{C}$, the *Hausdorff content* $M_h(E)$ is defined as $\inf \sum_{D \in \mathscr{S}} h(\mathrm{diam}\, D)$ where $\mathscr{S}$ runs over all countable coverings of E by open or closed discs. In case $h(r) = r^{\beta}$ for some $\beta > 0$, we write $M_h(E) = M^{\beta}(E)$, the *β-dimensional Hausdorff content* of E. The *lower β-dimensional Hausdorff content* $M^{\beta}_{*}(E)$ is defined as $\sup_h M_h(E)$ where $h(r)$ runs over all positive increasing functions such that $h(r) \leqq r^{\beta}$, $\frac{h(r)}{r^{\beta}} \to 0$ as $r \downarrow 0$.

A *dyadic square* is a square with side 2^m and corners of the form $r_j 2^m + i s_j 2^m$ for integers m, r_j, s_j. We can define contents as above, but using open dyadic squares instead of discs. The contents so defined are comparable to those defined above. The dyadic contents corresponding to M_h, M^{β}, and M^{β}_{*} are denoted m_h, m^{β}, and m^{β}_{*}, respectively.

The properties of Hausdorff contents and dyadic Hausdorff contents are set out in [2], [7], [12], [15].

§ 3. Individual function theorem

(3. 1) Let X be a compact subset of the plane, and let $\mathscr{R}(X)$ denote the algebra of all C^{∞} functions f such that f coincides with a rational function on some neighbourhood of X.

Vitushkin's individual function theorem [19], p. 172 states that a function f, continuous on $\mathbb{C}$, belongs to the uniform closure of $\mathscr{R}(X)$ on X if and only if there exists $\eta(d) \downarrow 0$ such that

$$\left|\int f \bar{\partial} \phi \, d\mathscr{L}^2\right| \leqq \eta(d(\phi)) \, N(\phi) \, \gamma(D(\phi) \sim X)$$

for all $\phi \in \mathscr{D}$. Here γ is analytic capacity. Melnikov [12], Theorem 1 established that the analogue of γ for the Lip α norm is the content $M^{1+\alpha}$, so the analogue of Vitushkin's theorem is as follows.

Theorem 1. *Let* $0 < \alpha < 1$, *and let* $X \subset \mathbb{C}$ *be compact. Let* $f \in \operatorname{lip} \alpha$. *Then the following conditions are equivalent.*

(1) *For each* $\varepsilon > 0$, *there exists* $g \in \mathscr{R}(X)$ *such that*

$$\|f - g\|_{\alpha} < \varepsilon.$$

(2) *For each* $\varepsilon > 0$, *there exists* $g \in \mathscr{R}(X)$ *such that*

$$\|f - g\|_{\alpha, X} < \varepsilon.$$

(3)

$$\left|\int f \bar{\partial} \phi \, d\mathscr{L}^2\right| \leqq K_1 N(\phi) M^{1+\alpha}(\mathrm{D}(\phi) \sim X) \|f\|_{\alpha, D(\phi)},$$

for all $\phi \in \mathscr{D}$. *Here* K_1 *is a certain constant, depending only on* α.

(4) *There exists* $\eta(d) \downarrow 0$ *such that*

$$\left|\int f \bar{\partial} \phi \, d\mathscr{L}^2\right| \leqq \eta(d(\phi)) N(\phi) M^{1+\alpha}(D(\phi) \sim X)$$

for all $\phi \in \mathscr{D}$.

(3. 2) Obviously, (3) implies (4), and (1) implies (2).

The equivalent of (1) and (2) follows from the fact that there is a continuous extension map from lip(α, X) to lip α [18], ch. VI.

It remains to prove that (1) implies (3) and that (4) implies (1). First we need an estimate for the T_{ϕ} operator.

§ 4. Estimate for T_ϕ

(4. 1) The following estimate plays a central role in all our results.

Lemma. *Suppose $\omega(r)$ is a modulus such that*

$$\text{(4. 1. 1)} \qquad \frac{r \log r}{\omega(r)} \to 0 \quad \text{as} \quad r \downarrow 0.$$

Let $d_0 > 0$ be given. Then there exixts a constant $K_2(\omega, d_0) > 0$ such that

$$\| T_\phi f \|^\omega \leqq K_2 N(\phi) \| f \|^\omega$$

whenever $f: \mathbb{C} \to \mathbb{C}$ and $\phi \in \mathscr{D}$, with $d(\phi) \leqq d_0$.

(4. 2) In many cases, the constant K_2 does not depend on d_0. This is the case, in particular, when $\omega(r) = r^\alpha$, $0 < \alpha < 1$.

The case $\omega(r) = r^\alpha$ of the lemma was used in [14], and was stated without proof. We have used weaker forms of the estimate in a number of papers. It goes back originally to Bers [21]. The feature which is important for our present work is the explicit form of the dependence of the operator norm of T_ϕ on ϕ. There are some subtle points in the estimate, so we present the key points of the proof. Some of the ideas are due to Dolženko [4].

(4. 3) *Proof of Lemma.* Fix $x \neq y$, and let $r = |x - y|/2$.

We may assume that f vanishes at some point of spt ϕ, since $T_\phi 1 = 0$, so that neither side of the desired inequality is altered by the addition of a constant to f.

Let $d = d(\phi) \leqq d_0$. Then

$$\text{(4. 3. 1)} \qquad \| f\chi \|_\infty \leqq \omega(d) \| f \|^\omega$$

where χ is the characteristic function of spt ϕ. We have

$$|(T_\phi f)(x) - (T_\phi f)(y)| = \left| \phi(x) f(x) - \phi(y) f(y) - \frac{1}{\pi} \int\left\{ \frac{x - y}{(\zeta - x)(\zeta - y)} \right\} f(\zeta) \bar{\partial} \phi(\zeta) \, d\mathscr{L}^2(\zeta) \right|$$

$$\leqq |\phi(x) f(x) - \phi(y) f(y)| + \frac{2r}{\pi} \| f\chi \|_\infty \| \nabla \phi \|_\infty \int_{\operatorname{spt}\phi} \frac{d\mathscr{L}^2(\zeta)}{|\zeta - x|\,|\zeta - y|} = A + B, \text{ say.}$$

Case 1. $r > d$. Then one of x, y is distant at least $r/2$ from spt ϕ, so we obtain

$$\begin{aligned} A &\leqq \| \phi \|_\infty \| f\chi \|_\infty \\ &\leqq \| \phi \|_\infty \, \omega(d) \, \| f \|^\omega, \quad \text{by (4. 3. 1)} \\ &\leqq \| \phi \|_\infty \| f \|^\omega \, \omega(2r), \quad \text{since } \omega \uparrow, \\ B &\leqq \frac{2r}{\pi} \| f \|^\omega \, \omega(2r) \, \| \nabla \phi \|_\infty \frac{2}{r} \int_{D(\phi)} \frac{d\mathscr{L}^2(\zeta)}{|\zeta - z|}, \end{aligned}$$

where z is x or y. The latter integral is estimated by $2\pi d$, so

$$B \leqq 8d \, \| \nabla \phi \|_\infty \| f \|^\omega \, \omega(2r),$$

$$A + B \leqq 8 N(\phi) \| f \|^\omega \, \omega(2r).$$

Case 2. $r \leqq d$. Since ω is a modulus,

$$\frac{2r}{\omega(2r)} \cdot \frac{\omega(d)}{d} \leqq \frac{2d}{\omega(2d)} \cdot \frac{\omega(2d)}{d} = 2,$$

so we have

$$A \leqq \|\phi\|_\infty \; \|f\|^\omega \, \omega(2r) + \|f\chi\|_\infty \; |\phi(x) - \phi(y)|$$
$$\leqq \|\phi\|_\infty \; \|f\|^\omega \, \omega(2r) + \omega(d) \; \|f\|^\omega \; \|\nabla\phi\|_\infty \, 2r$$

(by (4. 3. 1) and the mean value theorem)

$$\leqq 2'\{\|\phi\|_\infty + d \; \|\nabla\phi\|_\infty\} \; \|f\|^\omega \, \omega(2r).$$

Let D_x and D_y denote the closed discs of radius $7r/4$, centred at x and y respectively. Then $D_x \cup D_y$ contains the closed disc D_w of radius $5r/4$ centred at the point $w = (x+y)/2$. Let E denote the part of $\operatorname{spt}\phi$ outside D_w. Then $\operatorname{spt}\phi \subset D_x \cup D_y \cup E$, so we may estimate

$$\int_{\operatorname{spt}\phi} \frac{d\mathscr{L}^2(\zeta)}{|\zeta - x| \, |\zeta - y|} \leqq \int_{D_x} + \int_{D_y} + \int_E = I_1 + I_2 + I_3, \text{ say.}$$

Now,

$$I_1 \leqq \frac{4}{r} \int_{D_x} \frac{d\mathscr{L}^2(\zeta)}{|\zeta - x|} = 7\pi,$$

and similarly, $I_2 \leqq 7\pi$. For $\zeta \in E$ we have

$$|\zeta - w| \leqq 5 \min\{|\zeta - x|, |\zeta - y|\},$$

hence

$$I_3 \leqq 25 \int_E \frac{d\mathscr{L}^2(\zeta)}{|\zeta - w|^2} \leqq 50\pi \int_r^{3d} \frac{dr}{r} \leqq 100\pi + 50\pi \log\left(\frac{d}{r}\right),$$

$$B \leqq \left\{228 + 100 \log\left(\frac{d}{r}\right)\right\} \cdot r \cdot \omega(d) \cdot \|f\|^\omega \cdot \|\nabla\phi\|_\infty \cdot$$

Consider the function $p(r,d)$, defined on the closed triangle $T = \{(r,d) : 0 \leqq r \leqq d \leqq d_0\}$ by

$$p(r,d) = \begin{cases} 0 & , \; r = 0, \\ \dfrac{\omega(d)}{d} \cdot \dfrac{r}{\omega(r)} \cdot \log\left(\dfrac{d}{r}\right), & r > 0. \end{cases}$$

The hypotheses on ω ensure that $p(r,d)$ is continuous, and hence bounded, on T. Let $K_3(\omega, d)$ be the supremum of p on T. Then

$$B \leqq \{228 + 100 \, K_3\} \; d \; \|\nabla\phi\|_\infty \; \|f\|^\omega \, \omega(2r),$$
$$A + B \leqq (230 + 100 \, K_3) \; N(\phi) \; \|f\|^\omega \, \omega(2r).$$

Thus in both cases,

$$|(T_\phi f)(x) - T_\phi f)(y)| \leqq K_2 \; N(\phi) \; \|f\|^\omega \, \omega(|x-y|),$$

where $K_2 = 230 + 100 \, K_3$. The lemma is proved.

§ 5. Proof of theorem 1

(5. 1) (1) ⇒ (3). By continuity, it suffices to prove that (3) holds for all $f \in \mathscr{R}(X)$. The left-hand side is the absolute value of $(T_\phi f)'(\infty)$. But $T_\phi f$ is analytic off $D(\phi) \sim X$, hence by an estimate of Dolženko [4],

$$|(T_\phi f)'(\infty)| \leqq K_4 \|T_\phi f\|_\alpha M^{1+\alpha}(D(\phi) \sim X)$$

where K_4 depends only on α. The result now follows from the lemma.

(5. 2) (4) ⇒ (1). This is a routine application of known technique. In [14] the Vitushkin-Davie approximation scheme [19], [3] was modified to apply to one case of Lip α approximation. The modified scheme is not adequate to deal with all possible questions of Lip α approximation by analytic functions, since it leans on the metric character of the capacities in the case studied. However, the scheme will work whenever the capacities are Hausdorff contents. For this reason the scheme in [14], § 15 can be applied to prove the implication (4) ⇒ (1). We omit the details.

§ 6. Analytically negligible sets

(6. 1) A compact set $E \subset \mathbb{C}$ is said to be *analytically negligible* if every function f, continuous on the sphere and analytic on an open set U in the sphere Σ can be approximated uniformly on Σ by functions which are analytic on U and on a neighbourhood of E [6], p. 234. It is conjectured that every compact null-set for continuous analytic capacity is analytically negligible. This would follow from the subadditivity of continuous analytic capacity. Our next result is that the analogue of this conjecture holds for Lip α approximation.

Theorem 2. *Let E be a compact subset of $\mathbb{C}$, and let $0 < \alpha < 1$. The following are equivalent.*

(1) *If $f \in \operatorname{lip} \alpha$ and is analytic on an open set U, and $\varepsilon > 0$ is given, then there exists a function $g \in \operatorname{lip} \alpha$ such that g is analytic on U and on a neighbourhood of E, and*

$$\|f - g\|_\alpha < \varepsilon.$$

(2) *If $f \in \operatorname{lip} \alpha$ and is analytic on $\Sigma \sim E$, then f is constant.*

(3) $M_*^{1+\alpha}(E) = 0$.

(4) *There exists $\kappa > 0$ such that*

$$M_*^{1+\alpha}(F \sim E) \geqq \kappa M_*^{1+\alpha}(F)$$

for every closed set $F \subset \mathbb{C}$.

(6. 2) *Proof.* Obviously (1) ⇒ (2).

The implication (2) ⇒ (3) is essentially a theorem of Dolženko [4], 7, p. 75. Suppose $M_*^{1+\alpha}(E) > 0$. Then there exists an increasing function $h(r)$ such that $h(r) \leqq r^{1+\alpha}$, $h(r)/r^{1+\alpha} \to 0$ as $r \downarrow 0$, and $M_h(E) > 0$. We may assume that $h(r) = r\omega(r)$ where ω is a modulus. Furthermore, by [14], Lemma 4 we may assume that ω is a *modulated function* in the sense of [14], p. 190. This guarantees the hypotheses of Dolženko's theorem, hence there exists a nonconstant function f, analytic off E, such that $\|f\|^\omega < \infty$. This functions belongs to lip α, since $\omega(r)/r^\alpha \to 0$.

It is obvious that (3) $\Rightarrow$ (4), since $M_*^{1+\alpha}$ is subadditive.

The meat of the theorem is the implication (4) $\Rightarrow$ (1). In order to tackle this, we use two capacities. Fix an open set U. For any closed disc D, we say that a function $f \in \operatorname{lip}\alpha$ is D-1-admissible if $\|f\|_\alpha \leqq 1$ and f is analytic on U and on $\Sigma \sim D$. We say that f is D-2-admissible if it is D-1-admissible and is analytic on a neighbourhood of E. We define

$$\gamma_j(D) = \sup\{|f'(\infty)|: \text{ is } D\text{-}j\text{-admissible}\} \ (j=1,2).$$

By [14], p. 192; Cor. 7, there is a constant $K_5 > 0$, depending only on α, such that

$$K_5^{-1}\gamma_1(D) \leqq M_*^{1+\alpha}(D \sim U) \leqq K_5 \gamma_1(D)$$

for every closed disc D.

We claim there is another constant $K_6 > 0$ such that

$$K_6^{-1}\gamma_2(D) \leqq M_*^{1+\alpha}(D \sim U \sim E) \leqq K_6 \gamma_2(D)$$

for all closed discs D.

Assuming this claim for the moment, both γ_1 and γ_2 are essentially Hausdorff contents, hence the approximation scheme of [14] applies, and shows that (4) $\Rightarrow$ (1).

To prove the claim, note first that [14], Cor. 7 implies that $\gamma_2(D)$ is comparable to

$$\sup\{M_*^{1+\alpha}(D \sim U \sim N): N \text{ is a neighbourhood of } E\}.$$

Thus it comes down to showing that there is a constant $K_7 > 0$ such that

$$\lim_n M_*^{1+\alpha}(F_n) \geqq K_7\, M_*^{1+\alpha}(F)$$

whenever $F_n \uparrow F$. Carleson [2], p. 9 (3. 2) proved that

$$\lim_n m_h(F_n) = m_h(F)$$

for any increasing function h. This implies that

$$\lim_n M_h(F_n) \geqq K_7\, M_h(F)$$

where K_7 is a universal constant. The claim follows, by the definition of $M_*^{1+\alpha}$. The theorem is proved.

It is worth remarking that the implication (4) $\Rightarrow$ (3) is trivial. Simply choose $F = E$.

§ 7. Comparisons between norms

(7. 1) Let $0 < \alpha < \beta < 1$. We turn now to approximation of and by lip β analytic functions, in Lip α norm. The first step is to estimate the appropriate capacity, and for this we must see how the Lip β norm of an analytic function controls its Lip α norm. Our goal is the estimate (7. 8). We break the proof into a series of lemmas.

(7. 2) Lemma. *Let $\omega(r)$ be a modulus. Let f analytic off a disc D of diameter d, with $f(\infty) = 0$. Then*

$$\|f\|_\infty \leqq \sqrt{2}\, \omega(d)\, \|f\|^\omega.$$

Proof. Let $f=u+iv$, with u and v real-valued. By the maximum principle u and v each have zeros on bdy D. Thus, on D,

$$|u| \leqq \omega(d)\,\|v\|^{\omega} \leqq \omega(d)\,\|f\|^{\omega},$$
$$|v| \leqq \omega(d)\,\|v\|^{\omega} \leqq \omega(d)\,\|f\|^{\omega},$$
$$|f| \leqq \sqrt{2}\,\omega(d)\,\|f\|^{\omega},$$

and the result follows from the maximum principle.

(7. 3) Lemma. *Let* $0<\alpha<1$, *and let* $\omega(r)$ *be a modulus with*

$$\eta(r)=\sup\{\omega(s)/s^{\alpha}: 0<s\leqq r\}\downarrow 0.$$

Let f *be analytic off a disc* D *of diameter* d, *with* $f(\infty)=0$. *Then*

$$\|f\|_{\alpha} \leqq 2\sqrt{2}\,\eta(d)\,\|f\|^{\omega}.$$

Proof. Fix $x \neq y$ in $\mathbb{C}$.

Case 1. $|x-y|\leqq d$. Then

$$|f(x)-f(y)| \leqq \omega(|x-y|)\,\|f\|^{\omega} \leqq \eta(d)\,\|f\|^{\omega}\,|x-y|^{\alpha}.$$

Case 2. $|x-y|>d$. Then

$$|f(x)-f(y)| \leqq 2\|f\|_{\infty} \leqq 2\sqrt{2}\,\omega(d)\,\|f\|^{\omega}, \text{ by Lemma (7. 2)}$$
$$\leqq 2\sqrt{2}\,\eta(d)\,\|f\|^{\omega}\,|x-y|^{\alpha}.$$

The result follows.

(7. 4) Lemma. *Let* $\omega(r)$ *be a modulus, let* $h(r)=r\omega(r)$, *let* f *be analytic off a compact set* E *and vanish at* ∞. *Let* $z\notin E$, *and set* $d=\operatorname{dist}(z,E^{*})$, *where* E^{*} *is the smallest closed disc containing* E. *Then*

$$|f(z)| \leqq \frac{2\sqrt{2}}{\pi}\cdot\frac{M_h(E)\,\|f\|^{\omega}}{d}, \tag{7. 4. 1}$$

$$|f'(z)| \leqq \frac{2\sqrt{2}}{\pi}\cdot\frac{M_h(E)\,\|f\|^{\omega}}{d^2}. \tag{7. 4. 2}$$

This result is proved in the same fashion as [14], Lemma 12. Just cover E by squares and use the Cauchy integral theorem.

(7. 5) Lemma. *Let* $h(r)$ *be continuous, increasing and positive for* $r>0$, *with* $h(0)=0$. *Let* E *be any subset of* $\mathbb{C}$. *Let* p *be a positive integer. Let*

$$s=h^{-1}(M_h(E)).$$

Let $\{B_n\}$ *be a countable covering of* E *by closed discs of radius* s, *such that no point belongs to more than* p *of the* B_n*'s. Then there exists a universal constant* K_8, *such that*

$$\sum_n M_h(B_n\cap E) \leqq K_8\,p\,M_h(E).$$

We note that h maps $\mathbb{R}^{+}=\{r: r\geqq 0\}$ homeomorphically onto im$h$, and $M_h(E)\in$ im$h$ (by the definition of $M_h(E)$), hence $h^{-1}(M_h(E))$ exists. This lemma generalises the first assertion of [14], Lemma 13, and the same proof works.

(7. 6) Lemma. *Let $\omega(r)$ be a modulus satisfying* (4. 1. 1), *and let $h(r)=r\omega(r)$. Let E be a compact subset of the plane with* $\operatorname{diam} E \leqq 10$. *Let f be analytic off E, with $f(\infty)=0$. Then*

$$\|f\|_\infty \leqq K_9 \cdot \omega(h^{-1}(M_h(E))) \cdot \|f\|^\omega$$

where $K_9>0$ depends only on ω.

Proof. Cover E by discs $\{B_j'\}$ of diameter $s=h^{-1}(M_h(E))$, in such a way that no point belongs to more than 4. Let B_j be the disc concentric with B_j', with twice the radius. Choose functions $\phi_j \in \mathcal{D}$ with $\operatorname{spt}\phi_j \subset B_j$, $0 \leqq \phi_j \leqq 1$, $\sum \phi_j = 1$ on a neighbourhood of $\bigcup B_j'$, and $\|\nabla \phi_j\|_\infty \leqq 4/s$. Let $f_j = T_{\phi_j} f$. Then $f=\sum f_j$, and f_j is analytic off B_j. By Lemma (4. 1),

$$\|f_j\|^\omega \leqq K_{10} \|f\|^\omega, \tag{7. 6. 1}$$

where K_{10} depends only on ω.

Fix $x \in \mathbb{C}$. Divide the integers j into classes G_m $(m=0, 1, 2, \dots)$ by the rule: $j \in G_m$ if and only if m is the greatest nonnegative integer such that $\operatorname{dist}(z, B_j) \geqq ms$. There are at most $K_{11} \max\{m, 1\}$ integers in the class G_m. We have

$$\begin{aligned}
|f(x)| &\leqq \sum_{j \in G_0} |f_j(x)| + \sum_{m=1}^{\infty} \sum_{j \in G_m} |f_j(x)| \\
&\leqq \sqrt{2} K_{11} \cdot \omega(2s) \cdot \|f_j\|^\omega + \sum_{m=1}^{\infty} \sum_{j \in G_m} \frac{2\sqrt{2}}{\pi} \cdot \frac{M_h(B_j \cap E) \cdot \|f_j\|^\omega}{ms}, \text{ by (7. 2) and (7.4.1)} \\
&\leqq K_{12} \cdot \|f\|^\omega \cdot \omega(s) \cdot \left\{1 + \sum_{m=1}^{\infty} \sum_{j \in G_m} \frac{M_h(B_j \cap E)}{m h(s)}\right\}, \text{ by (7. 6. 1)} \\
&\leqq K_{13} \|f\|^\omega \omega(s) \left\{1 + \sum_j \frac{M_h(B_j \cap E)}{h(s)}\right\}^{\frac{1}{2}} \text{ (cf. [6], (2. 6), p. 201)} \\
&\leqq K_9 \|f\|^\omega \omega(s), \text{ by Lemma (7. 5).}
\end{aligned}$$

The lemma is proved.

(7. 7) Lemma. *Let* $0<\alpha<1$. *Let $\omega(r)$ be a modulus satisfying* (4. 1. 1), *and such that* $\omega(r)/r^\alpha \to 0$, *and let $h(r)=r\omega(r)$. Let*

$$\eta(r) = \sup\{\omega(s)/s^\alpha : 0 < s \leqq r\}.$$

Let E and f be as in (7. 6). *Then*

$$\|f\|_\alpha \leqq K_{15} \cdot \eta(h^{-1}(M_h(E))) \cdot \|f\|^\omega.$$

Proof. Choose s, B_j, ϕ_j, f_j as in (7. 6). Fix $x, y \in \mathbb{C}$. There are two cases to consider.

Case 1. $|x-y| \geqq s$. Then

$$\begin{aligned}
|f(x)-f(y)| &\leqq 2\|f\|_\infty \leqq 2K_9\, \omega(s)\, \|f\|^\omega, \text{ by (7. 6),} \\
&\leqq 2K_9\, s^{-\alpha} \omega(s) \cdot \|f\|^\omega \cdot |x-y|^\alpha.
\end{aligned}$$

Case 2. $|x-y| \leqq s$. Classify the integers j into classes H_m ($m=0, 1, 2, \ldots$) according to the rule: $j \in H_m$ if and only if m is the greatest nonnegative integer such that

$$ms \leqq \operatorname{dist}\left(\frac{x+y}{2}, B_j\right).$$

Then there are at most $K_{14} \max\{m, 1\}$ integers in H_m, and we have

$$|f(x)-f(y)| \leqq \sum_{j\in H_0} |f_j(x)-f_j(y)| + \sum_{m=1}^{\infty} \sum_{j\in H_m} |f'_j(z_j)|\,|x-y|$$

(where z_j is some point between x and y)

$$\leqq K_{14}\, K_{16}\, \eta(2s) \cdot \|f\|^{\omega} \cdot |x-y|^{\alpha} + \sum_{m=1}^{\infty} \sum_{j\in H_m} \frac{K_{16} \cdot M_h(B_j \cap E) \cdot \|f\|^{\omega} \cdot s^{1-\alpha} \cdot |x-y|^{\alpha}}{m^2 s^2}$$

(by (4. 1), (7. 3), and (7. 4. 2))

$$\leqq K_{17} \cdot \eta(s) \cdot \|f\|^{\omega} \cdot |x-y|^{\alpha} \cdot \left\{1 + \sum_{m=1}^{\infty} \sum_{j\in H_m} \frac{M_h(B_j \cap E)}{m^2 h(s)}\right\}$$

$$\leqq K_{18} \cdot \eta(s) \cdot \|f\|^{\omega} \cdot |x-y|^{\alpha}, \text{ as before.}$$

The lemma follows.

(7. 8) Corollary. *Let* $0<\alpha<\beta<1$. *Let* E *be a compact subset of the plane, and let* f *be analytic off* E, *with* $f(\infty)=0$. *Then*

$$\|f\|_{\alpha} \leqq K_{15} \cdot M^{1+\beta}(E)^{\frac{\beta-\alpha}{1+\beta}} \cdot \|f\|_{\beta},$$

where K_{15} *depends only on* α *and* β.

The restriction on the diameter of E is not needed in this case.

§ 8. Approximating smoother functions

(8. 1) Let $B(x, r)$ denote the closed disc with centre x and radius r.

Theorem 3. *Let* $0<\alpha<\beta<1$, *and let* X *be a compact subset of* $\mathbb{C}$. *Then the following conditions are equivalent.*

(1) *Each function* $f \in \operatorname{Lip}\beta$ *which is analytic on* $\operatorname{int} X$ *is the limit in* $\operatorname{Lip}(\alpha, X)$ *norm of a sequence of rational functions with poles off* X.

(2) *There exists a constant* $\kappa>0$ *such that*

$$M^{1+\alpha}(D \sim X)^{\frac{1}{1+\alpha}} \geqq \kappa\, M^{1+\beta}(D \sim \operatorname{int} X)^{\frac{1}{1+\beta}}$$

for every closed disc D.

(3) $$\limsup_{r\downarrow 0} \frac{M^{1+\alpha}(B(x, r) \sim X)}{r^{1+\alpha}} > 0 \quad \textit{for } M^{1+\beta} \textit{ almost all } x \in \operatorname{bdy} X.$$

The equivalence (1) $\Leftrightarrow$ (3) is analogous to Melnikov's theorem [12], Theorem 3. The uniform analogue of the equivalence (1) $\Leftrightarrow$ (2) states:

Each function $f \in \mathrm{Lip}\,\beta$ which is analytic on $\mathrm{int}\,X$ is a uniform limit on X of rational functions if and only if there exists $\kappa > 0$ such that

$$\gamma(D \sim X) \geqq \kappa M^{1+\beta}(D \sim \mathrm{int}\ X)^{\frac{1}{1+\beta}}$$

for every closed disc M, where γ is analytic capacity.

This is also true, in view of the density theorem [12], Lemma 4, and Melnikov's theorem.

Theorem 4. *Let $0 < \alpha < \beta < 1$, and let X be a compact subset of $\mathbb{C}$. Then the condition*

(1) *there exists $\kappa > 0$ such that*

$$M^{1+\beta}(D \sim \mathrm{int}\ X)^{\frac{1}{1+\beta}} \geqq \kappa M_*^{1+\alpha}(D \sim \mathrm{int}\ X)^{\frac{1}{1+\alpha}}$$

for every closed disc D,

implies the condition

(2) *each function $f \in \mathrm{lip}\,\alpha$ which is analytic on $\mathrm{int}\,X$ is a limit in $\mathrm{lip}\,\alpha$ norm of functions in $\mathrm{Lip}\,\beta$ which are analytic on $\mathrm{int}\,X$.*

There is an analoguous theorem for approximation by functions in lip β, with $M^{1+\beta}$ replaced by $M_*^{1+\beta}$.

(8. 2) Fix $0 < \alpha < \beta < 1$, and a compact set $X \subset \mathbb{C}$. We shall use three capacities to prove these theorems. We say a function $f \in \mathrm{lip}\,\alpha$ with $\|f\|_\alpha \leqq 1$, analytic off a closed disc D is D-1-admissible (respectively, D-2-admissible, respectively, D-3-admissible) if f is analytic on a neighbourhood of X (respectively, analytic on int X, respectively, in Lip β and analytic on int X). We define

$$\gamma_j(D) = \sup\{|f'(\infty)| : f \text{ is } D\text{-}j\text{-admissible}\}$$

for $j = 1, 2, 3$ (these γ_j are not to be confused with those defined in § 6). Then, as we know, $\gamma_1(D)$ is comparable to $M^{1+\alpha}(D \sim X)$, and $\gamma_2(D)$ is comparable to $M_*^{1+\alpha}(D \sim \mathrm{int}\,X)$.

Lemma. *There exists a constant K_{19}, depending only on α and β, such that*

$$\gamma_3(D) \geqq K_{19} M^{1+\beta}(D \sim \mathrm{int}\ X)^{\frac{1+\alpha}{1+\beta}}$$

for all closed discs D.

Proof. Fix a closed disc D, and let $E = D \sim \mathrm{int}\,X$. By Melnikov's theorem [12], Theorem 1 there exists a function $f \in \mathrm{Lip}\,\beta$ such that f is analytic off E, $\|f\|_\beta \leqq 1$, and $f'(\infty) \geqq K_{20} M^{1+\beta}(E)$, where K_{20} depends only on β. By Lemma (7. 8), the function

$$K_{15}^{-1} \cdot M^{1+\beta}(E)^{\frac{\alpha-\beta}{1+\beta}} \cdot f$$

is D-3-admissible, and the result follows.

(8. 3) *Proof of Theorems* 3 *and* 4. The capacities γ_1 and γ_2 are comparable to contents, and γ_3 is bounded below by a content, so the approximation scheme of [14] proves Theorem 4 and the implication (1) $\Rightarrow$ (2) of Theorem 3.

The implication (2) $\Rightarrow$ (3) follows from the density theorem [7], (4. 4), p. 121, which states that for any Borel set E,

$$\limsup_{r\downarrow 0} \frac{M^{1+\beta}(B(x,r)\cap E)}{r^{1+\beta}} \geqq \frac{1}{25}$$

for $M^{1+\beta}$ allmost all $x\in E$.

It remains to prove the implication (3) $\Rightarrow$ (1). The proof is suggested by the proof of sufficiency in [12], Theorem 3. First, we need an inequality.

Lemma. *Let* $0<\gamma<\delta$ *and let* $E\subset \mathbb{C}$. *Then*

$$M^\delta(E)^{\frac{1}{\delta}} \leqq M^\gamma(E)^{\frac{1}{\gamma}}.$$

Proof. Let $\{D_n\}$ be a countable covering of E by discs, with diam $D_n=d_n$. Then

$$\{\sum_n d_n^\delta\}^{\frac{\gamma}{\delta}} \leqq \sum_n d_n^\gamma$$

since the function $t\longrightarrow t^{\gamma/\delta}$ $(t>0)$ is concave. The lemma follows.

Now assume (3) holds. Let $f\in \operatorname{Lip}\beta$ be analytic on the interior of X. Then by Dolženko's estimate, Lemma (4. 1), and Lemma (7. 8),

$$|\int f\,\partial\phi\, d\,\mathscr{L}^2| \leqq K_{21}\cdot N(\phi)\cdot M^{1+\beta}(D(\phi)\sim \operatorname{int} X)\cdot \|f\|_\beta$$

for each $\phi\in\mathscr{D}$. To show that f is approximable by rationals, it suffices, by Theorem 1, to prove that

$$M^{1+\beta}(D\sim \operatorname{int} X) \leqq \varepsilon(d)\, M^{1+\alpha}(D\sim X)$$

for all closed discs D, where $d=\operatorname{diam} D$, and $\varepsilon(d)\downarrow 0$ as $d\downarrow 0$. We shall in fact show that

$$M^{1+\beta}(D\sim \operatorname{int} X) \leqq K_{22}\, d^{\beta-\alpha}\, M^{1+\alpha}(D\sim X).$$

Fix a closed disc D, with diameter d. There are two cases to consider.

Case 1. $M^{1+\beta}(D\sim X) \geqq M^{1+\beta}(D\cap \operatorname{bdy} X)$. We have

$$\begin{aligned} M^{1+\beta}(D\sim \operatorname{int} X) &\leqq M^{1+\beta}(D\sim X) + M^{1+\beta}(D\cap \operatorname{bdy} X) \leqq 2\, M^{1+\beta}(D\sim X) \\ &\leqq 2\, d^{\beta-\alpha}\, M^{1+\beta}(D\sim X)^{\frac{1+\alpha}{1+\beta}} \leqq 2\, d^{\beta-\alpha}\, M^{1+\alpha}(D\sim X), \end{aligned}$$

where we have used Lemma (8. 2).

Case 2. $M^{1+\beta}(D\sim X) < M^{1+\beta}(D\cap \operatorname{bdy} X)$. Arguing as in [12], p. 122 we find a countable family of pairwise-disjoint squares $\{S_n\}$, with sides $r_n<2d$ such that

$$M^{1+\alpha}(S_n\sim X) \geqq d^{\alpha-\beta}\cdot r_n^{1+\beta},$$

$$\sum_n r_n^{1+\beta} \geqq K_{23}\, M^{1+\beta}(D \cap \text{bdy } X)$$

and such that for every square S,

$$\sum_{S_n \subset S} r_n^{1+\beta} \leqq 2\, r^{1+\beta}$$

where r is the side of S. We choose functions $f_n \in \mathscr{R}(X)$ such that f_n is analytic off S_n, $f_n(\infty)=0$, $f_n'(\infty)=r_n^{1+\beta}$, and $\|f_n\|_\alpha \leqq d^{\beta-\alpha}$. Then $f=\sum_n f_n$ satisfies $\|f\|_\alpha \leqq K_{24}\, d^{\beta-\alpha}$ (this is proved as in [12], p. 119), and we have

$$M^{1+\beta}(D \sim \text{int } X) \leqq 2\, M^{1+\beta}(D \cap \text{bdy } X) \leqq 2\, K_{23}^{-1} \sum_n r_n^{1+\beta}$$

$$= 2\, K_{23}^{-1} f'(\infty) \leqq K_{25}\, d^{\beta-\alpha}\, M^{1+\alpha}(D \sim X)$$

by Dolženko's estimate.

This concludes the proof of Theorem 3.

§ 9. Further results

The following theorem may be proved by the approximation scheme of [14], using [14], Cor. 7 and Carleson's theorem as in (6. 2) to identify the capacities.

Theorem 5. *Let* $0<\alpha<1$. *Let* X *be a compact subset of* $\mathbb{C}$, *and let* E *and* F *be closed subsets of* bdy X. *Then the following conditions are equivalent.*

(1) *Each function* $f \in \text{lip}\, \alpha$ *which is analytic on* int X *and on a neighbourhood of* E *is a* Lip(α, X) *limit of functions in* lip α *which are analytic on* int X *and on a neighbourhood of* F.

(2) *There exists a constant* $\kappa>0$ *such that*

$$M_*^{1+\alpha}(D \sim F \sim \text{int } X) \geqq \kappa\, M_*^{1+\alpha}(D \sim E \sim \text{int } X)$$

for each closed disc D.

There are a number of partial results along these lines which may be obtained by juggling with Lip β conditions and analyticity on various sets. We can also get one direction of an individual function theorem for Lip α approximation by Lip β analytic functions, by applying Lemma (8. 2).

Finally, a problem. The argument of (8. 3) suggests an "instability" result ([19], pp. 288—290) might hold for the content $M^{1+\alpha}$. Obviously, such a result must be purely measure-theoretic, and valid in all dimensions. Prove or disprove the following:

Let $E \subset \mathbb{R}^n$ *be a Borel set, and let* $0<\alpha<\beta<\infty$. *Then for* M^β *almost all* x, *either*

$$\limsup_{r \downarrow 0} \frac{M^\alpha(B(x,r) \cap E)}{r^\alpha} \geqq \frac{1}{4^n}$$

or else

$$\lim_{r \downarrow 0} \frac{M^\alpha(B(x,r) \cap E)}{r^\beta} = 0.$$

References

[1] *T. Bagby*, Quasi topologies and rational approximation, J. Functional Analysis **10** (1972), 259—268.
[2] *L. Carleson*, Selected Problems on Exceptional Sets, Wokingham, Berkshire 1967.
[3] *A. M. Davie*, Analytic capacity and approximation problems, Transactions AMS **171** (1972), 409—444.
[4] *E. P. Dolžeňko*, The removability of singularities of analytic functions (Russian), Uspehi Mat. Nauk **18** **(112)** (1963), 135—142.
[5] *W. H. J. Fuchs*, Théorie de l'Approximation des Fonctions d'une Variable Complexe, Montreal 1968.
[6] *T. W. Gamelin*, Uniform Algebras, Prentice Hall 1969.
[7] *J. Garnett*, Analytic Capacity and Measure, LNM **297**, Berlin-Heidelberg-New York 1972.
[8] *A. A. Gonchar* and *S. N. Mergelyan*, Uniform approximation by analytic and harmonic functions (Russian), Proceedings Erevan Conference, Moscow 1966, 94—101.
[9] *V. P. Havin*, Approximation in the mean by analytic functions, Soviet Math. Doklady **6** (1965), 1458—1460.
[10] *L. I. Hedberg*, Approximation in the mean by analytic functions, Transactions AMS **163** (1972), 157—171.
[11] *L. I. Hedberg*, Non-linear potentials and approximation in the mean by analytic functions, Math. Zeit. **129** (1972), 299—319.
[12] *M. S. Melnikov*, Metric properties of analytic α-capacity and approximation of analytic functions with a Hölder condition by rational functions, Math. USSR Sbornik **8** (1969), 115—124.
[13] *A. G. O'Farrell*, Localness of certain Banach modules, Indiana Univ. Math. J. **24** (1975), 1135—1141.
[14] *A. G. O'Farrell*, Hausdorff content and rational approximation in fractional Lipschitz norms, Transactions AMS **228** (1977), 187—206.
[15] *A. G. O'Farrell*, Continuity properties of Hausdorff content, J. London Math. Soc. (2) **13** (1976), 403—410.
[16] *S. O. Sinanjan*, Approximation by polynomials and analytic functions in the areal mean, AMS Translations **(2) 74** (1968), 91—124.
[17] *S. O. Sinanjan*, Completeness of analytic functions and extremality of the coefficients in the Laurent series (Russian), Mat. Zam. **22** (1977), 277—283.
[18] *E. M. Stein*, Singular Integrals and Differentiability Properties of Functions, Princeton 1970.
[19] *A. G. Vitushkin*, The analytic capacity of sets in problems of approximation theory, Russian Math. Surveys **22** no. **6** (1967), 139—200.
[20] *L. Zalcman*, Analytic Capacity and Rational Approximation, LNM **50**, Berlin-Heidelberg-New York 1968.
[21] *L. Bers*, Theory of pseudo-analytic functions, NYU, 1953.

Maynooth College, Co. Kildare, Ireland

Eingegangen 20. September 1978