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RATIONAL APPROXIMATION IN LIPSCHITZ NORMS—II

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ABSTRACT

We consider rational approximation in Lip β norm, for non-integral $\beta > 1$, by functions analytic and rational on a neighbourhood of an arbitrary fixed compact set in the plane. We distinguish three natural problems concerning approximation of this type, and we solve two of them. We show further that for a compact set with no interior, all lip β functions are approximable by rationals if and only if the set is a subset of a finite disjoint union of lip β curves. We also consider approximation by elements of function modules over the rationals, and by solutions of elliptic partial differential equations.

This paper is about approximation in Lip β norm, for non-integral $\beta > 1$, by functions analytic and rational on a neighbourhood of an arbitrary fixed compact set in the plane. We distinguish three natural problems concerning approximation of this type, and we solve two of them. The main results on approximation are Theorems 1, 2, and 3. Besides standard methods of functional analysis, we use the Whitney extension technique [16, Chapter VI], Geometric structure theory [4, section (3.5); 7], point derivations on Banach algebras [15, 7, 10, 12, 13], and the Cauchy transform, in the spirit of [9, 10]. Theorem 1 has analogues for approximation by elements of function modules over the rationals (Theorem 1′), and for approximation by solutions of elliptic partial differential equations (Theorem 1″).

1. Formulation of the problems

For $0 < d \in \mathbb{Z}$ and $0 \leqslant m \in \mathbb{Z}$, let $C^m(\mathbb{R}^d)$ denote the space of bounded continuous complex-valued functions on $\mathbb{R}^d$ with bounded continuous partial derivatives up to order m. For $0 < \alpha \leqslant 1$, let Lip $(\alpha, \mathbb{R}^d)$ denote the space of functions $f \in C^0(\mathbb{R}^d)$ such that there exists $\kappa > 0$ with $|f(x) - f(y)| \leqslant \kappa |x-y|^\alpha$ for all points x and y belonging to $\mathbb{R}^d$. Let lip $(\alpha, \mathbb{R}^d)$ denote the space of all those functions $f \in$ Lip $(\alpha, \mathbb{R}^d)$ such that, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| \leqslant \varepsilon |x-y|^\alpha$ whenever $|x-y| < \delta$. Of course, lip $(1, \mathbb{R}^d)$ contains only the constant functions. For $\beta = m + \alpha$, with $0 < m \in \mathbb{Z}$ and $0 < \alpha \leqslant 1$, let Lip $(\beta, \mathbb{R}^d)$ (respectively, lip $(\beta, \mathbb{R}^d)$) denote the space of functions in $C^m(\mathbb{R}^d)$ with all m-th order partial derivatives belonging to Lip $(\alpha, \mathbb{R}^d)$

(respectively, lip $(\alpha, \mathbb{R}^d)$). Let $\Xi(d, m)$ denote the set of multi-indices $j = (j_1, \ldots, j_d)$ with $j_k \geqslant 0$ and $|j| = j_1 + \ldots + j_d \leqslant m$, and set $j! = j_1!\, j_2! \ldots j_d!$. The partial derivative corresponding to j we denote by D_j.

The spaces $C^0(\mathbb{R}^d)$, $C^m(\mathbb{R}^d)$, Lip $(\alpha, \mathbb{R}^d)$, and Lip $(\beta, \mathbb{R}^d)$ become Banach algebras under pointwise multiplication, when endowed with the norms

$$\|f\|_{C^0} = \sup\{|f(x)| : x \in \mathbb{R}^d\},$$
$$\|f\|_{C^m} = \sum_{j \in \Xi(d,m)} (j!)^{-1} \|D_j f\|_{C^0},$$
$$\|f\|_\alpha = \|f\|_{C^0} + \text{least } \kappa,$$
$$\|f\|_\beta = \|f\|_{C^{m-1}} + \sum_{|j| = m} \|D_j f\|_\alpha.$$

The space lip $(\beta, \mathbb{R}^d)$ is a closed subalgebra of Lip $(\beta, \mathbb{R}^d)$ with respect to the norm $\|*\|_\beta$.

Let $\mathscr{D}(\mathbb{R}^d)$ denote the space of infinitely-differentiable complex-valued functions on $\mathbb{R}^d$ with compact support.

Let $0 < m \in \mathbb{Z}$, and let $a \in \mathbb{R}^d$. Then for any function $\gamma : \Xi(d, m) \to \mathbb{C}$, the formula

$$Lf = \sum_{|j| \leq m} \gamma_j D_j f(a) \qquad (f \in C^m(\mathbb{R}^d))$$

defines an element L of the dual $C^m(\mathbb{R}^d)^*$. Such an L we call an m-th *order bounded point differential operator at the point* a.

Let X be a compact subset of $\mathbb{R}^d$, and define $I(X) = \{f \in C^0(\mathbb{R}^d) : f = 0 \text{ on } X\}$ Then $I(X) \cap C^m(\mathbb{R}^d)$ is a closed ideal in $C^m(\mathbb{R}^d)$. For $a \in X$ we define $J^m(X, a)$ as the space of all m-th order point differential operators L at the point a, such that $Lf = 0$ whenever $f \in I(X) \cap C^m(\mathbb{R}^d)$. We define the m-*th order tangent bundle of* X as

$$J^m(X) = \cup\{J^m(X, a) : a \in X\}.$$

We identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. For $0 \leqslant n \in \mathbb{Z}$, we let $\bar{\mathscr{P}}_n$ denote the $(n + 1)$-dimensional space of conjugate-analytic polynomials $\alpha_0 + \alpha_1 \bar{z} + \ldots + \alpha_n \bar{z}^n$ on $\mathbb{C}$. For compact $X \subset \mathbb{C}$, we let $\mathscr{R}(X)$ denote the algebra of all functions

$$f \in \bigcap_{m=1}^{\infty} C^m(\mathbb{C})$$

such that there exists a rational function g with poles off X such that $f = g$ on a neighbourhood of X. The product $\mathscr{R}(X)\bar{\mathscr{P}}_n$ forms an $\mathscr{R}(X)$-module. These modules were studied in [9, 10].

We will use the abbreviations C^m, Lip β, and lip β only for $C^m(\mathbb{C})$, Lip $(\beta, \mathbb{C})$ and lip $(\beta, \mathbb{C})$, respectively.

Let $\beta > 0$, let X be a compact subset of $\mathbb{C}$, and let $f \in \text{Lip } \beta$. Consider the following statements about f.

(1) There exists a sequence of functions $f_n \in \mathscr{R}(X)$ such that $\|f - f_n\|_\beta \to 0$.

(2) There exists a function $g \in I(X) \cap \text{lip } \beta$ and a sequence $f_n \in \mathscr{R}(X)$ such that $\|f - g - f_n\|_\beta \to 0$.

(3) There exists a sequence of functions $f_n \in \mathscr{R}(X) + (I(X) \cap \text{lip } \beta)$ such that $\|f - f_n\|_\beta \to 0$.

Note that f must belong to lip β if (1) or (2) holds. Clearly, (1) implies (2), and (2) implies (3). For $k = 1, 2, 3$, we use the expression, "*the* k-th *problem of* Lip β *rational approximation*", to mean the problem of deciding for which pairs (X, f) statement (k) is true. In case $0 < \beta < 1$, the three problems are equivalent, and have been solved completely [8]. If $\beta \geqslant 1$, then the first and second problems are not equivalent. In case $\beta = 1$, the second and third problems are equivalent; this follows from the theorem of [13]. It is not clear whether or not the second and third problems have the same solution in general. We cannot expect any progress on this question until we understand more about the structure of compact subsets of $\mathbb{R}^d$, from the point of view of local embeddings in C^m submanifolds, for $m \geqslant 2$.

2. Solution of the first problem of Lip β rational approximation

Let $\bar{\partial}$ denote the differential operator $\partial/\partial x + i\,\partial/\partial y$ $(= D_{(1,0)} + i\,D_{(0,1)})$ on $C^1(\mathbb{C})$. The following theorem is a solution to the first problem, for nonintegral $\beta > 1$.

Theorem 1. *Le X be a compact subset of $\mathbb{C}$. Let $\beta = m + \alpha$, where $1 \leqslant m \in \mathbb{Z}$ and $0 < \alpha < 1$. Let $f \in$ lip β. Then the following statements are equivalent.*

(*A*) *There exists a sequence of functions $f_n \in \mathscr{R}(X)$ such that $\|f - f_n\|_\beta \to 0$.*

(*B*) *The function $\bar{\partial} f$ vanishes on X, together with all its partial derivatives up to order $m - 1$, that is, $D_j \bar{\partial} f = 0$ on X for $|j| \leqslant m - 1$.*

We shall in fact prove the following more general theorem.

Theorem 1′. *Let X, β, m, α, and f be as above, and let $0 \leqslant r \in \mathbb{Z}$ with $r \leqslant m$. Then the following statements are equivalent.*

(*A*) *There exists a sequence of functions $f_n \in \mathscr{R}(X)\overline{\mathscr{P}}_r$ such that $\|f - f_n\|_\beta \to 0$.*

(*B*) $D_j(\bar{\partial})^{r+1} f = 0$ *on X whenever $|j| \leqslant m - r - 1$.*

PROOF. Clearly (*A*) implies (*B*). To prove the converse, suppose (*B*) holds, and let $T \in (\text{lip } \beta)^*$ be an annihilator of $\mathscr{R}(X)\overline{\mathscr{P}}_r$. We wish to show that Tf must vanish. This will give the result, by the separation theorem.

Fix, once and for all, a function $\psi \in \mathscr{D}(\mathbb{C})$ such that $\psi = 1$ on a neighbourhood of X. For functions $g \in \text{lip}\,\alpha$ with compact support, we define the Cauchy transform $\hat{g}$ by setting

$$\hat{g}(z) = \frac{1}{\pi}\int \frac{g(\tau)}{z-\tau}\,d\mathscr{L}^2(\tau)$$

whenever $z \in \mathbb{C}$. Here $\mathscr{L}^2$ denotes Lebesgue measure on $\mathbb{C}$. For the properties of this operator, see [10]. The most important facts are that $\hat{g} \in \text{lip}\,(1+\alpha)$, and $\widehat{\bar{\partial}g} = g$. We define the operator $C : \text{lip}\,\alpha \to \text{lip}\,(1+\alpha)$ by setting $Cg = \widehat{\psi g}$. For each nonnegative integer n, the operator C maps $\text{lip}\,(n+\alpha)$ continuously into $\text{lip}\,(n+1+\alpha)$ [10, p. 381, Lemma 6]. Thus the adjoint C^* maps $\text{lip}\,(n+1+\alpha)^*$ into $\text{lip}\,(n+\alpha)^*$. For $n \geqslant 1$ and $g \in \text{lip}\,(n+\alpha)$, with compact support, we have $(\bar{\partial}g)^\wedge = g$. Taking $g = \psi f$, we obtain $\psi f = [\bar{\partial}(\psi f)]^\wedge = (\psi\bar{\partial}f)^\wedge + (f\bar{\partial}\psi)^\wedge = C\,\bar{\partial}f + (f\,\bar{\partial}\psi)^\wedge$. Since $f\bar{\partial}\psi = 0$ on a neighbourhood of X, we have $(f\,\bar{\partial}\psi)^\wedge$ analytic on a neighbourhood of X, hence by the general Runge theorem [10, p. 375], $(f\,\bar{\partial}\psi)^\wedge$ is approximable in Lip β norm by elements of $\mathscr{R}(X)$. Thus $T(f\,\bar{\partial}\psi)^\wedge = 0$. Also, $f - \psi f \in \mathscr{R}(X)$, so $Tf = T\psi f = TC\,\bar{\partial}f = (C^*T)\,\bar{\partial}f$.

If $h \in \text{lip}\,(\beta-1)$ and $h = 0$ on a neighbourhood of X, then $(\psi h)^\wedge$ is analytic on a neighbourhood of X, hence $(C^*T)h = TCh = 0$. Thus the support of C^*T is contained in X.

If $r = 0$, we stop. Otherwise, suppose $h \in \mathscr{R}(X)\bar{\mathscr{P}}_{r-1}$. Then $\psi h \in \mathscr{R}(X)\bar{\mathscr{P}}_{r-1}$, so $(C^*T)h = T(\psi h)^\wedge = 0$, by the Key Lemma of [10, p. 375]. Thus C^*T annihilates $\mathscr{R}(X)\bar{\mathscr{P}}_{r-1}$, and we may repeat the above procedure, to get

$$Tf = [C^*C^*T][\bar{\partial}\bar{\partial}f].$$

Continuing, we get

$$Tf = [(C^*)^{r+1}T][(\bar{\partial})^{r+1}f],$$

where $S = (C^*)^{r+1}\,T$ belongs to $\text{lip}\,(\beta - r - 1)^*$, and the support of S is contained in X. The result now follows from the next lemma.

Lemma 1. *Let $\beta = m + \alpha$, $0 \leqslant m \in \mathbb{Z}$, $0 < \alpha < 1$, let $X \subset \mathbb{R}^d$ be compact, let $f \in \text{lip}\,(\beta, \mathbb{R}^d)$, and suppose $D_j f(a) = 0$ whenever $|j| \leqslant m$ and $a \in X$. Then there exists a sequence $f_n \in \text{lip}\,(\beta, \mathbb{R}^d)$ such that each f_n vanishes on a neighbourhood of X, and $\|f - f_n\|_\beta \to 0$.*

Proof. Suppose f satisfies the hypotheses, and let $\varepsilon > 0$ be given. Then there exists a neighbourhood V_1 of X such that $|D_j f(a)| < \varepsilon$ whenever $a \in V_1$ and $|j| \leqslant m$. For each j belonging to $\Xi(d, m)$, the function F_j, defined by

$$F_j(x,y) = \begin{cases} \dfrac{D_j f(x) - \sum\limits_{|k| \leq m-|j|} \dfrac{(x-y)^k\, D_{j+k}f(y)}{k!}}{|x-y|^{m-|j|+\alpha}}, & x \neq y \\ 0, \quad x = y \end{cases}$$

is continuous on $\mathbb{R}^d \times \mathbb{R}^d$, and vanishes on $X \times X$; here $(x-y)^k$ denotes the product $(x_1-y_1)^{k_1} \ldots (x_d-y_d)^{k_d}$. Hence there exists a closed neighbourhood $V_2 \subset V_1$ of X in $\mathbb{R}^d$ such that $|F_j(x, y)| < \varepsilon$ for x and y belonging to V_2 and $|j| \leqslant m$. By the Whitney-Calderón-Zygmund extension theorem [16, Chapter VI] there exists a function $g \in \text{lip}\,(\beta, \mathbb{R}^d)$ such that $f = g$ on V_2 and $\|g\|_\beta < \kappa\varepsilon$, where κ is a certain constant that depends only on β and d. Thus the function $h = f - g$ vanishes on V_2 and satisfies $\|f-h\|_\beta < \kappa\varepsilon$. The proof is complete.

It is well established by now that there is a close analogy between rational approximation and approximation by solutions of elliptic partial differential equations on $\mathbb{R}^d$ [1, 2, 5, 6, 11, 14, 17]. Some parts of this analogy are not yet well understood, but in many cases theorems go over routinely from one setting to the other. Theorem 1 falls in to the latter category and the analogous result is as follows.

Theorem 1″. *Let X be a compact subset of $\mathbb{R}^d$, let $L(D)$ be a constant-coefficient elliptic operator on $\mathbb{R}^d$, of order $p \geqslant 2$, let $\beta = m + \alpha$, where $p \leqslant m \in \mathbb{Z}$, and $0 < \alpha < 1$, and let $f \in \text{lip}\,(\beta, \mathbb{R}^d)$. Then the following statements are equivalent.*

(*A*) *There exists a sequence of functions $f_n \in \text{lip}\,(\beta, \mathbb{R}^d)$ such that $L(D)f_n = 0$ on a neighbourhood of X and $\|f-f_n\|_\beta \to 0$.*

(*B*) *$D_j L(D)f = 0$ on X whenever $|j| \leqslant m-p$.*

PROOF. There exists a locally-integrable kernel $K(x, y)$ such that, defining

$$(Pf)(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,dy,$$

we have $L(D)Pf = f = PL(D)f$ whenever $f \in \mathscr{D}(\mathbb{R}^d)$. Moreover, the operator P maps lip $(\gamma, \mathbb{R}^d)$ continuously into lip $(p + \gamma, \mathbb{R}^d)$ whenever $0 < \gamma \notin \mathbb{Z}$ [11]. Using the operator P in place of C^{r+1}, the argument of Theorem 1′ is easily modified to prove the present result.

Theorems 1, 1′, and 1″ also have analogues in the theory of approximation in the Sobolev spaces $W^{m,p}(\mathbb{C})$ and $W^{m,p}(\mathbb{R}^d)$, for $1 \leqslant m \in \mathbb{Z}$ and $1 < p < \infty$, where $W^{m,p}(\mathbb{R}^d)$ denotes the space of functions f in $L^p(\mathbb{R}^d)$ such that all partial derivatives (in the sense of distributions) up to order m are represented by functions in $L^p(\mathbb{R}^d)$. The statements of the analogues are obvious, and the proofs carry over because the Calderón-Zygmund estimates for the kernels also work in the context of $W^{m,p}$ spaces (indeed, it was for these spaces that Calderón and Zygmund first proved them [3]).

3. Solution of the second problem of Lip β rational approximation

Let X be a compact subset of $\mathbb{C}$, let $\beta = m + \alpha$, where $1 \leqslant m \in \mathbb{Z}$ and $0 < \alpha < 1$, and let $f \in \text{lip}\,\beta$. In view of Theorem 1, we may formulate the second problem as follows: Give conditions on the pair (X, f) that guarantee the existence of a function $f^* \in \text{lip}\,\beta$ such that $f^* = f$ on X and $D_j\,\bar{\partial} f^* = 0$ on X whenever $|j| \leqslant m-1$. Thus the second problem reduces to an extension problem.

Suppose f^* exists as above. Suppose

$$L = \sum_{|j| \leq m} \alpha_j D_j(a)$$

belongs to the m-th order tangent bundle $J^m(X)$, where $\alpha_j = \alpha_j(L) \in \mathbb{C}$ and $a = a(L) \in X$. Then $Lf = Lf^*$. Moreover,

$$D_j f^*(a) = \frac{\partial^{j_1}}{\partial x^{j_1}} \frac{\partial^{j_2}}{\partial y^{j_2}} f^*(a) = (-i)^{j_2} \frac{\partial^{|j|}}{\partial x^{|j|}} f^*(a),$$

$$Lf^* = \sum_{k=0}^{m} \beta_k \frac{\partial^k f^*(a)}{\partial x^k} ,$$

where

$$\beta_k = \beta_k(L) = \sum_{|j|=k} (-i)^{j_2} \alpha_j.$$

Thus, Lf depends only on $a(L)$ and on the values at L of the functions $\beta_0, \beta_1, \ldots, \beta_m : J^m(X) \to \mathbb{C}$. This gives one necessary condition for the existence of f^*. Furthermore, if we define $G(a, \beta_0, \ldots, \beta_m)$ as Lf^*, where $L \in J^m(X)$ has $a, \beta_0, \ldots, \beta_m$ as parameters, then G is defined on a certain subset of $X \times \mathbb{C}^{m+1}$. For $0 \leqslant k \leqslant m$, define $G_k(a)$ as the value of $G(a, \beta_0, \ldots, \beta_m)$, when all the β_r except β_k are set equal to 0, and β_k is set equal to 1, provided this value exists. Then $G_0(a)$ is defined on all of X, and $G_k(a)$, for $1 \leqslant k \leqslant m$, is defined on X', the set of accumulation points of X. These functions satisfy the following condition:

$$(4) \quad \left| G_k(a) - \sum_{r=0}^{m-k} \frac{G_{k+r}(b)(a-b)^r}{r!} \right| \leqslant \varepsilon(\delta) \, |a-b|^{m-k+\alpha}$$

whenever $0 \leqslant k \leqslant m$, $a \in X$, $b \in X'$, $G_k(a)$ is defined, $\delta > 0$, and $|a-b| < \delta$. Here $(a-b)^r$ denotes the r-th power of the complex number $a-b$, and $\varepsilon(\delta)$ is a positive function of δ, that tends to zero as δ tends to zero. This gives us a second necessary condition for the existence of f^*. The next theorem solves the second problem by showing that these two conditions are sufficient.

Theorem 2. *Let X be a compact subset of $\mathbb{C}$, let $\beta = m + \alpha$, $0 < m \in \mathbb{Z}$, $0 < \alpha < 1$, and let $f \in \operatorname{lip}(\beta, \mathbb{C})$ be given. Then the following conditions are equivalent.*

(*A*) *There exists a function $f^* \in \operatorname{lip} \beta$ such that $f = f^*$ on X and $D_j \bar{\partial} f^* = 0$ on X whenever $|j| \leqslant m-1$.*

(*B*) *The function $G : J^m(X) \to \mathbb{C}$ defined by $GL = Lf$ depends only on $a(L)$, $\beta_0(L), \ldots, \beta_m(L)$, and the associated functions $G_k(a)$ satisfy the condition* (4) *whenever all the terms are defined.*

Proof. We have already shown that (*A*) implies (*B*).

Conversely, suppose (*B*) holds. Choose a Whitney cowering [16, Chapter VI; 12] of $\mathbb{C} \sim X'$ by cubes Q_n ($n = 1, 2, 3, \ldots$) with the side of Q_n comparable to its distance from X', and with the property that no point belongs to more than 100^d of the Q_n's.

Choose a partition of unity $\{\phi\}^{\infty}{}_1 \subset \mathscr{D}(\mathbb{C})$ subordinate to $\{Q_n\}^{\infty}{}_1$, and choose points $p_n \in X'$ such that p_n is at least as close to Q_n as any other point of X'. For $a \in X \sim X'$, and $1 \leqslant k \leqslant m$, define

$$G_k(a) = \sum_{n=1}^{\infty} \phi_n(a) \left\{ \sum_{r=0}^{m-k} \frac{G_{k+r}(p_n)(a-p_n)^r}{r!} \right\}.$$

Then $G_k(a)$ is defined on all of X for $0 \leqslant k \leqslant m$, and a routine check shows that, for all $\delta > 0$,

$$\left| G_k(a) - \sum_{r=0}^{m-k} \frac{G_{k+r}(b)(a-b)^r}{r!} \right| \leqslant \varepsilon_1(\delta)\, |a-b|^{m-k+\alpha}$$

whenever $a \in X$, $b \in X$, and $|a-b| < \delta$, where $\varepsilon_1(\delta)$ tends to zero as δ tends to zero. Thus, if we define

$$h_j(a) = (i)^{j_2} G_{|j|}(a)$$

for $j \in \Xi(2, m)$ and $a \in X$, then we see that the collection $\{h_j\}$ satisfies the hypotheses of the Whitney-Calderón-Zygmund extension theorem, hence there exists a function $f^* \in \operatorname{lip} \beta$ such that $D_j f^* = h_j$ on X, for $j \in \Xi(2, m)$. Hence, if $|j| \leqslant m-1$ and $a \in X$, we have

$$\begin{aligned} D_j \bar{\partial} f^*(a) &= D_j \frac{\partial f^*}{\partial x}(a) + i\, D_j \frac{\partial f^*}{\partial y}(a) \\ &= h_{(j_1+1, j_2)}(a) + i h_{(j_1, j_2+1)}(a) \\ &= (i)^{j_2} G_{|j|+1}(a) + i(i)^{j_2+1} G_{|j|+1}(a) \\ &= 0. \end{aligned}$$

Thus (A) holds. This completes the proof.

It is reasonable to ask for a description of those compact sets $X \subset \mathbb{C}$ with the property that *all* functions $f \in \operatorname{lip} \beta$ may be approximated by rationals on X, in the sense of the second problem. In view of Theorem 2, this amounts to asking for a characterisation of those X such that, for all $f \in \operatorname{lip} \beta$,

(i) Lf depends only on $(a, \beta_0, \ldots, \beta_m)$ for $L \in J^m(X)$, and

(ii) the associated functions G_k satisfy condition (4).

We shall provide a simple answer to this question in Theorem 3. First we need a few lemmas. These lemmas are special cases of the meta-theorem that the inverse of a function is as smooth as the function itself, except near critical points.

Lemma 2. *Let A and B be compact subsets of $\mathbb{C}$. Let f_0 be a homeomorphism of A onto B, such that*

$$\kappa^{-1} |a_1 - a_2| \leqslant |f_0(a_1) - f_0(a_2)| \leqslant \kappa\, |a_1 - a_2|$$

for all pairs a_1, a_2 of points of A, where the constant κ does not depend on a_1 or a_2.

Let g_0 be the inverse of f_0. Let $0 \leqslant m \in \mathbb{Z}$ and $0 \leqslant \alpha < 1$. Suppose there exist continuous functions $f_1, \ldots, f_m : A' \to \mathbb{C}$, and a positive function $\varepsilon_1(\delta) \downarrow 0$ such that

$$\left| f_0(a_1) - \sum_{r=0}^{n} \frac{f_r(a_2)(a_1 - a_2)^r}{r!} \right| \leqslant |a_1 - a_2|^{m+\alpha}\, \varepsilon_1(|a_1 - a_2|)$$

whenever $a_1 \in A$, and $a_2 \in A'$. Then there exist continuous functions $g_1, \ldots, g_m : B' \to \mathbb{C}$ and a positive function $\varepsilon_2(\delta) \downarrow 0$, such that

$$\left| g_0(b_1) - \sum_{r=0}^{m} \frac{g_r(b_2)(b_1 - b_2)^r}{r!} \right| \leqslant |b_1 - b_2|^{m+\alpha}\, \varepsilon_2(|b_1 - b_2|)$$

whenever $b_1 \in B$ and $b_2 \in B'$.

PROOF. The case $m = 0$ is trivial, so we suppose now that m is at least 1. Choose $M > 0$ such that $|f_k(a)| \leqslant M$ for $0 \leqslant k \leqslant m$ and $a \in A'$. For $a \in A'$, consider the polynomial

$$p(a, \zeta) = \sum_{k=1}^{m} \frac{f_k(a)}{k!} \zeta^k.$$

We have $|f_1(a)| \geqslant \kappa^{-1}$, so

$$|p(a, \zeta) - f_1(a)\zeta| \leqslant \sum_{k=2}^{m} \frac{M|\zeta|^k}{k!}$$

$$\leqslant M|\zeta|^2 e^{|\zeta|} < \kappa^{-1}|\zeta| \leqslant |f_1(a)\zeta|$$

provided $|\zeta| < R = \min\{\log 2, 1/2M\}$. Thus, by Rouché's theorem, $p(a, \zeta)$ is an invertible analytic function on the disc $|\zeta| < R$. For $1 \leqslant r \in \mathbb{Z}$, we define the functions $g_r : B' \to \mathbb{C}$ by requiring that the identity

$$\zeta = \sum_{r=1}^{\infty} \frac{g_r(b)}{r!} \left\{ \sum_{k=1}^{m} \frac{f_k(g_0(b))\zeta^k}{k!} \right\}^r$$

hold for all $b \in B'$ and all ζ with $|\zeta| < R$. This identity yields recursion relations for the $g_r(b)$, in terms of the $f_k(g_0(b))$. The first few relations are:

$$1 = g_1 f_1,$$

$$0 = g_1 f_2 + g_2 f_1^2,$$

$$0 = g_1 f_3 + 2g_2 f_1 f_2 + g_3 f_1^3,$$

$$0 = g_1 f_4 + g_2 f_2^2 + 2g_2 f_1 f_3 + 3g_3 f_1^2 f_2 + g_4 f_1^4.$$

The n-th relation is:

$$0 = g_1 f_n + g_2 \sum_{j_1+j_2=n} f_{j_1} f_{j_2} + \ldots + g_k \sum_{j_1+\cdots+j_k=n} f_{j_1} \ldots f_{j_k} + \ldots + g_n f_1^n$$

where $f_k = 0$, for $k > m$. Thus $g_r(b)$ is a rational function of $f_1(a), \ldots, f_r(a)$ (where $a = g_0(b)$), in which the denominator may be taken as a power of f_1. Thus, all the g_r are continuous on B', since $f_1, \ldots, f_m$ are continuous on A', and $|f_1|$ is bounded below. In particular, the functions $g_1, \ldots, g_m$ are continuous. Furthermore, if $b_1 = f_0(a_1)$ and $b_2 = f_0(a_2)$, with $a_1 \in A$ and $a_2 \in A'$, then for small $|b_1 - b_2|$ we have

$$\begin{aligned} g(b_1) - g(b_2) &= a_1 - a_2 \\ &= \sum_{r=1}^{\infty} \frac{g_r(b_2)}{r!} \left\{ \sum_{k=1}^{m} \frac{f_k(a_2)(a_1 - a_2)^k}{k!} \right\}^r \\ &= \sum_{r=1}^{\infty} \frac{g_r(b_2)}{r!} \left\{ f(a_1) - f(a_2) + |a_1 - a_2|^{m+\alpha} o(|a_1 - a_2|) \right\}^r \\ &= \sum_{r=1}^{\infty} \frac{g_r(b_2)(b_1 - b_2)^r}{r!} + |a_1 - a_2|^{m+\alpha} o(|a_1 - a_2|) \\ &= \sum_{r=1}^{m} \frac{g_r(b_2)(b_1 - b_2)^r}{r!} + |b_1 - b_2|^{m+\alpha} o(|b_1 - b_2|). \end{aligned}$$

This proves the lemma.

Lemma 3. *Let $A, B, f_k, g_r, m, \alpha$, and $\varepsilon_1(\delta)$ be as in Lemma 2. Suppose that in addition the f_k satisfy the condition:*

$$\left| f_k(a_1) - \sum_{r=0}^{m-k} \frac{f_{k+r}(a_2)(a_1 - a_2)^r}{r!} \right| \leqslant |a_1 - a_2|^{m-k+\alpha} \varepsilon_1(|a_1 - a_2|),$$

whenever $1 \leqslant k \leqslant m$, $a_1 \in A'$, and $a_2 \in A'$. Then the functions g_r satisfy the corresponding condition:

$$\left| g^k(b_1) - \sum_{r=0}^{m-k} \frac{g_{k+r}(b_2)(b_1 - b_2)^r}{r!} \right| \leqslant |b_1 - b_2|^{m-k+\alpha} \varepsilon_3(|b_1 - b_2|),$$

whenever $1 \leqslant k \leqslant m, b_1 \in B'$, and $b_2 \in B'$.

PROOF. We note that $g_k(b) = h^{(k)}(b, b)$ (i.e., the k-th derivative with respect to ω of $h(b, \omega)$, evaluated at $\omega = b$), where $h(b, \omega)$ is the inverse function of $p(g_0(b), \zeta)$ with $h(b, 0) = 0$. Thus, for b_1 and b_2 belonging to B', we have

$$\begin{aligned} &\left| h^{(k)}(b_2, b_1) - \sum_{r=0}^{m-k} \frac{g_{k+r}(b_2)(b_1 - b_2)^r}{r!} \right| \\ &= \left| h^{(k)}(b_2, b_1) - \sum_{r=0}^{m-k} \frac{h^{(k+r)}(b_2, b_2)(b_1 - b_2)^r}{r!} \right| \\ &\leqslant M_1 |b_1 - b_2|^{m-k+1} \end{aligned}$$

for a suitable constant $M_1 = M_1(\kappa, M, \text{diam } B)$. Thus it suffices to prove that

$$|h^{(k)}(b_2, b_1) - h^{(k)}(b_1, b_1)| \leqslant |b_1 - b_2|^{m-k+\alpha}\varepsilon_4(|b_1 - b_2|),$$

for some $\varepsilon_4(\delta) \downarrow 0$. Let $a_j = g_0(b_j)$, for $j = 1, 2$. Then $h^{(k)}(b_2, b_1)/k!$ is the k-th coefficient in the inverse series of

$$\sum_{n=1}^{\infty} \frac{p^{(n)}(a_2, a_1)z^n}{n!} = \sum_{n=1}^{m} \left\{ \sum_{r=0}^{m-n} \frac{f_{n+r}(a_2)(a_1 - a_2)^r}{r!} \right\} \frac{z^n}{n!}$$

Also, $h^{(k)}(b_1, b_1)/k!$ is the k-th coefficient in the inverse series of

$$\sum_{n=1}^{\infty} \frac{p^{(n)}(a_1, a_1)\, z^n}{n!} = \sum_{n=1}^{\infty} f_n(a_1) \frac{z^n}{n!}$$

Thus $h^{(k)}(b_2, b_1)$ is a certain rational function of the numbers

$$\sum_{r=0}^{m-n} \frac{f_{n+r}(a_2)(a_1 - a_2)^r}{r!} \quad (n = 1, 2, \ldots, k)$$

and $h^{(k)}(b_1, b_1)$ is the *same* rational function of the numbers $f_n(a_1)$ $(n = 1, 2, \ldots, k)$. In view of the hypothesis, we deduce that

$$|h^{(k)}(b_2, b_1) - h^{(k)}(b_1, b_1)| \leqslant |a_1 - a_2|^{m-k+\alpha}\varepsilon_5(|a_1 - a_2|) \leqslant |b_1 - b_2|^{m-k+\alpha}\varepsilon_4(|b_1 - b_2|),$$

and the result follows.

Theorem 3. *Let X be a compact subset of $\mathbb{C}$, and let $\beta = m + \alpha$, $1 \leqslant m \in \mathbb{Z}$, $0 < \alpha < 1$. Then the following are equivalent.*

(A) *For each $f \in$ lip β and $L \in J^m(X)$, the value of*

$$Lf = \sum_{|j| \leq m} \alpha_j D_j f(a)$$

depends only on f, a, and the parameters $\beta_0, \ldots, \beta_m$, where

$$\beta_k = \sum_{|j| = k} (-i)^{j_2} \alpha_j.$$

Moreover, the associated functions $G_k(a)$ satisfy the condition (4).

(B) *X is a subset of a finite union of pairwise-disjoint simple* lip β *curves.*

Here, the expression "simple lip β curve" means the image of either the unit interval or the unit circle under a lip β diffeomorphism.

PROOF. It is clear that (B) implies (A).

Conversely suppose (A) holds. Fix $a \in X$. The argument in the proof of Theorem A of [7] shows that there exists a closed disc U about a in $\mathbb{C}$ and an orthogonal projection $\pi(x, y) = \lambda x + \mu y$ of $\mathbb{C} = \mathbb{R}^2 \to \mathbb{R}$ such that the restriction of π to $X \cap U$ is invertible with Lip 1 inverse. Let Y denote $\pi(X \cap U)$, and let $F_0 : Y \to X \cap U$ be the inverse of π.

We claim that there exist continuous functions $F_1, \ldots, F_m$ mapping $Y' \to \mathbb{C}$ such that

$$\left| F_k(t) - \sum_{r=0}^{m-k} \frac{F_{k+r}(u)(t-u)^r}{r!} \right| \leqslant |t-u|^{m-k+\alpha}\varepsilon(|t-u|)$$

whenever $0 \leqslant k \leqslant m$, $t \in Y$, $u \in Y'$, and $F_k(t)$ exists, where $\varepsilon(\delta) \downarrow 0$ as $\delta \downarrow 0$.

Given this, the Whitney-Calderón-Zygmund extension theorem implies that F has an extension $F^* \in \text{lip}\,(\beta,\mathbb{R})$, and it is easy to show that F^* may be modified, if necessary, to ensure that $dF^*/dt \neq 0$. Thus $X \cap U$ is a subset of a lip β arc. So each point a of X has a neighbourhood U such that $X \cap U$ is a subset of a lip β arc. This implies (*B*), as may be seen by imitating the argument of [7, pp. 162–163].

To prove the claim, first note that it is equivalent to the existence of functions $H_1, \ldots, H_m$ on $X' \cap U$ such that

$$\left| H_k(z) - \sum_{r=0}^{m-k} \frac{H_{k+r}(w)(\pi(z) - \pi(w))^r}{r!} \right| \leqslant |z-w|^{m-k+\alpha}\varepsilon(|z-w|)$$

for $z \in X \cap U$, $w \in X' \cap U$, $0 \leqslant k \leqslant m$, with $H_k(z)$ defined, where $H_0(z) = z$ and $\varepsilon(\delta) \downarrow 0$ as $\delta \downarrow 0$. Next, note that the function $f = \pi$ belongs to lip β, so that, by (*A*), the associated functions G_k satisfy condition (4). It is now clear that Lemmas 2 and 3 apply, with f_k replaced by G_k, and that these lemmas yield the desired functions H_k

It is worth noting that, in condition (4), the validity of the estimate for *even k*. $0 \leqslant k \leqslant m$, implies its validity for *odd k*, $0 < k \leqslant m$. This is proved by routine, calculation.

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