POINT DERIVATIONS ON AN ALGEBRA OF LIPSCHITZ FUNCTIONS

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ABSTRACT

Let E be a closed subset of $\mathbb{R}^d$. Let $D^1(E)$ denote the closure in Lip $(1, E)$ of the space of global C^1 functions. We determine the structure of the space of bounded point derivations on the real Banach algebra $D^1(E)$. We improve upon our constructive version of Whitney's extension theorem. We characterise those sets E such that all functions in $D^1(E)$ have C^1 extensions. We give other applications of derivations, and some examples.

Summary

Let E be a closed subset of a Euclidean space $\mathbb{R}^d$. We denote by $C^1(E)$ the algebra of restrictions to E of global C^1 functions. We denote by $D^1(E)$ the closure of $C^1(E)$ in Lip $(1, E)$. (The notation $D^1(E)$ was used in [5, 6, 7] for the corresponding space of complex-valued functions.) We study the sheaves $J(C^1, E)$ and $J(D^1, E)$ of bounded point derivations on $C^1(E)$ and $D^1(E)$ respectively. We describe (2.7) $J(D^1, E)$ in terms of a sheaf Tan E which has a geometric definition. We prove (3.2) a stronger, more elementary version of the extension theorem of [8], and deduce (3.3) a description of $J(C^1, E)$ in terms of Tan E. Our main result (4.1) is that $D^1(E) = C^1(E)$ if and only if they have the same bounded point derivations and the norms of the derivations on the two spaces are comparable. Section 5 contains some examples, and Section 6 contains another application of derivations.

We assume familiarity with [8], and with Sherbert's results [9] concerning derivations on Lip 1 spaces.

1. Definitions and preliminaries

Let E be a fixed closed subset of $\mathbb{R}^d$. For functions $f : E \to \mathbb{R}$ we define

$$\|f\|_{0,E} = \sup_E |f|,$$

$$\|f\|'_{1,E} = \inf\{\kappa > 0 : |f(a) - f(b)| \leqslant \kappa |a - b|, \quad \text{for all } a, b \in E\},$$

$$\|f\|_{1,E} = \|f\|_{0,E} + \|f\|'_{1,E}$$

Let

$$\mathrm{Lip}\,(1, E) = \{f \in \mathbb{R}^E : \|f\|_{1,E} < \infty\}.$$

Then Lip (1, E) becomes a semi-simple commutative (real) Banach algebra with identity [1, 4] when endowed with pointwise addition and multiplication and the norm $\|\cdot\|_{1,E}$. Let C^1 denote the space of bounded continuous real-valued functions on $\mathbb{R}^d$ with bounded continuous first partial derivatives. The algebra C^1 becomes a Banach algebra when endowed with the norm

$$\|f\|_{C_1} = \|f\|_{0,\mathbb{R}^d} + \left\{ \sum_{j=1}^{d} \left\| \frac{\partial f}{\partial x_j} \right\|^2_{0,\mathbb{R}^d} \right\}^{1/2}.$$

We denote the gradient of f by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d} \right).$$

We denote the usual inner product on $\mathbb{R}^d$ by $\langle , \rangle$, so that the directional derivative of f at the point $a \in \mathbb{R}^d$ in the direction of the unit vector $u \in \mathbb{R}^d$ is $\langle u, \nabla f(a) \rangle$.

Let $C^1(E)$ denote the space of real-valued functions f on E such that there exists $f^* \in C^1$ with $f^* = f$ on E. Then $C^1(E)$ becomes a Banach algebra when endowed with the quotient norm

$$\|f\|_{C^1(E)} = \inf\{\|f^*\|_{C^1} : f^* \in C^1, f^* = f \text{ on } E\}.$$

Lemma 1.1. *For each closed set $E \subset \mathbb{R}^d$ we have $C^1(E) \subset$ Lip (1, E), and the inclusion map is a contraction.*

PROOF. Let $f \in C^1(E)$ and let $\epsilon > 0$ be given. Then f has an extension $f^* \in C^1$ such that

$$\|f^*\|_{C^1} < (1 + \epsilon)\, \|f\|_{C^1(E)}.$$

Let $a, b \in E$, and let Γ denote the straight line from a to b. Then

$$\begin{aligned} |f(b) - f(a)| &= |f^*(b) - f^*(a)| \\ &= \left| \int_\Gamma \langle dx, \nabla f^*(x) \rangle \right| \\ &\leqslant |b - a| \sup |\nabla f^*|. \end{aligned}$$

Thus,

$$\begin{aligned} \|f\|_{1,E} &\leqslant \|f\|_{0,E} + \sup |\nabla f^*| \\ &\leqslant \|f^*\|_{C^1} \\ &\leqslant (1 + \epsilon)\, \|f\|_{C^1(E)}. \end{aligned}$$

Since this holds for all $\varepsilon > 0$, we conclude that

$$\|f\|_{1,E} \leqslant \|f\|_{C^1(E)}.$$

The result follows.

Let $D^1(E)$ denote the closure of $C^1(E)$ in Lip (1, E). Then $D^1(E)$ is a sub-Banach algebra of Lip (1, E). For nice sets E, $D^1(E)$ coincides with $C^1(E)$.

We note that the global C^∞ functions are dense in C^1, in the C^1 norm, and hence, by (1.1), $D^1(E)$ may also be described as the closure in Lip (1, E) of the space of (restrictions of) global C^∞ functions. For this reason, $D^1(E) \otimes \mathbb{C}$ is the natural context in which to study Lip 1 rational approximation. This is why we first introduced this space [5, 6, 7].

For the remainder of this section, let $F(\cdot)$ denote one of the functors $C^1(\cdot)$, $D^1(\cdot)$, Lip $(1, \cdot)$. Thus $F(\cdot)$ is a contravariant functor from the category of closed subsets of $\mathbb{R}^d$ (with inclusions as morphisms) to the category of commutative Banach algebras (with bounded algebra homomorphisms).

Fix E, and write $F = F(E)$ for brevity. Let $\Sigma(F)$ denote the spectrum of F (the space of algebra homomorphisms from F onto $\mathbb{R}$, with the relative weak-star topology from F^*, the dual of F). Consider the map $\phi_F : E \to \Sigma(F)$, given by $\phi_F(a)f = f(a)$ for all $f \in F$ and all $a \in E$. Since F separates points on E, the function ϕ_F is one-to-one. Since the functions $f \in F$ are continuous on E, and $\Sigma(F)$ has the relative weak-star topology, it follows that ϕ_F is continuous. The following fact is well-known [9, (2.1)].

Lemma 1.2. *$\phi_F(E)$ is weak-star dense in $\Sigma(F)$.*

In case E is compact, it follows that $\phi_F(E)$ equals $\Sigma(F)$, and ϕ_F is a homeomorphism, so that we may identify E with $\Sigma(C^1(E))$, $\Sigma(D^1(E))$, and Σ (Lip (1, E)). In case E is unbounded, $\Sigma(F) \sim \phi_F(E)$ is non-empty and in fact is large and horrible. Fortunately, as will appear, this fringe of the spectrum does not concern us.

A *bounded point derivation* on F at a homomorphism $\Theta \in \Sigma(F)$ is an element $d \in F^*$ such that

$$d(fg) = \Theta(f)\, dg + \Theta(g)\, df$$

whenever $f, g \in F$. There are nonzero bounded point derivations at all accumulation points of $\Sigma(F)$, but we only consider homomorphisms $\Theta \in \phi_F(E)$. If $\Theta = \phi_F(a)$, we call d a derivation on F *at the point* a.

Let $a \in E$. We denote the space of all bounded point derivations on $F(E)$ at a by $J(F, E, a)$. It is easy to check that $J(F, E, a)$ is a weak-star closed linear subspace of F^*. The following lemma rests on the fact that $C^1 \subset F$.

Lemma 1.3. *Let $a \in E$, let $f \in F(E)$, and suppose $f = 0$ on a neighbourhood of a. Then $df = 0$ for every $d \in J(F, E, a)$.*

PROOF. Suppose $f = 0$ on a neighbourhood U of a. Choose $\phi \in C^1$ such that $\phi(a) = 1$ and $\phi = 0$ off U. Then $\phi \in F$ and $\phi \cdot f = 0$, so

$$0 = d(\phi f) = \phi(a)\, df + f(a)\, d\phi = df$$

for every $d \in J(F, E, a)$, as required.

We set

$$J(F, E) = \{(a, d) : a \in E, d \in J(F, E, a)\},$$

and we endow $J(F, E)$ with the relative weak-star topology from $\mathbb{R}^d \times F^*$. We may regard $J(F, E)$ as a sheaf of vectorspaces over E. If $E_1 \subset E_2$, then

$$F(E_2) \subset F(E_1)$$

(that is, $f \mid E_1 \in F(E_1)$ whenever $f \in F(E_2)$). Hence for each $a \in E_1$ there is a natural injection

$$J(F, E_1, a) \to J(F, E_2, a).$$

Lemma 1.4. *Suppose $a \in E_1 \subset E_2$, where E_1 and E_2 are closed subsets of $\mathbb{R}^d$. Suppose there is a neighbourhood U of a in $\mathbb{R}^d$ such that $E_1 \cap U = E_2 \cap U$. Then the natural map $J(F, E_1, a) \to J(F, E_2, a)$ is surjective.*

PROOF. Suppose $d \in J(F, E_2, a)$. Choose $\phi \in C^1$ such that $\phi = 0$ off U and $\phi = 1$ on a neighbourhood V of a. Then for $f \in F(E_1)$ we have $\phi \cdot f \in F(E_2)$, where $\phi f = 0$ on $E_2 \sim U$. (Check this for each F.)

Define $d_1 f = d(\phi f)$ for all $f \in F(E_1)$. We claim that $d_1 \in J(F, E_1, a)$, and that $d_1 \mid F(E_2) = d$.

To see this fix f and g belonging to $F(E_1)$. Then $\phi f g \in F(E_2)$ hence

$$d_1(fg) = d(\phi fg) = d(\phi f \phi g) \quad \text{(by Lemma 1.3)} =$$
$$f(a)\, d(\phi g) + g(a)\, d(\phi f) = f(a)\, d_1 g + g(a)\, d_1 f.$$

Thus $d_1 \in J(F, E_1, a)$.

Moreover if $f \in F(E_2)$, then $d_1 f = d(\phi f) = df$, by Lemma 1.3.

The above proof also shows that the map $J(F, E_1, a) \to J(F, E_2, a)$, which is obviously a contraction, is a bi-continuous map of Banach spaces. (This fact also follows from the open mapping theorem). In case $F = C^1$ or Lip $(1, \cdot)$, one can show (under the hypotheses of (1.4)) that this map is an isometry. In case $F = D^1$, it is not clear to me whether or not the map is necessarily an isometry.

Corollary 1.5. *Let E_1 and E_2 be closed subsets of $\mathbb{R}^d$. Let $a \in E_1 \cap E_2$, and suppose there is a neighbourhood U of a in $\mathbb{R}^d$ such that $E_1 \cap U = E_2 \cap U$. Then there is a natural norm-bicontinuous linear isomorphism from $J(F, E_1, a)$ onto $J(F, E_2, a)$.*

PROOF. Apply (1.4) to the inclusions $E_1 \cap E_2 \subset E_1$ and $E_1 \cap E_2 \subset E_2$.

In view of Corollary 1.5, all problems concerning $J(F, E)$ are local in character.

REMARK 1. The occurrence of an extraneous fringe on the spectrum of F can be avoided altogether by removing the word "bounded" from the definitions of C^1 and Lip $(1, E)$, and working with the resulting Frechet algebras instead of Banach algebras. There is little to choose between the two approaches. On the whole we retain more information by working with norms.

REMARK 2. The functor F has the following "localness property": *Let E be a compact subset of $\mathbb{R}^d$, and let $f: E \to \mathbb{R}$. Suppose that each point $a \in E$ has a closed neighbourhood U in E such that $f \mid U \in F(U)$. Then $f \in F(E)$.*

This result is not hard to prove. The word "compact" cannot be replaced by "closed", as simple examples show.

2. Spaces of derivations

The purpose of this section is to describe the structure of $J(D^1, E, a)$. As a preliminary step, we describe $J(C^1, E, a)$.

First consider $C^1 = C^1(\mathbb{R}^d)$. Fix $a \in \mathbb{R}^d$. The functions $g \in C^1$ such that g is affine on a neighbourhood of the point a are dense in C^1. Hence, each derivation $d \in J(C^1, \mathbb{R}^d, a)$ is determined by its action on the functions affine near a. By (1.3), d is determined by its action on the globally-affine functions. Thus there exists $u \in \mathbb{R}^d$ such that $df = \langle u, \nabla f(a)\rangle$ whenever $f \in C^1$. The map $J(C^1, \mathbb{R}^d, a) \to \mathbb{R}^d$ given by $d \to u$ is linear, one-to-one, onto, and continuous (recall that $J(C^1, \mathbb{R}^d, a)$ has the relative weak-star topology from C^{1*}), and hence is a homeomorphism. Thus the sheaf $J(C^1, \mathbb{R}^d)$ is homeomorphic to the product sheaf $\mathbb{R}^d \times \mathbb{R}^d$. A cross section of $J(C^1, \mathbb{R}^d)$ on an open set $U \subset \mathbb{R}^d$ may be thought of as a first-order partial differential operator of the form

$$\sum_{j=1}^{d} g_j(x) \frac{\partial}{\partial x_j}$$

where the $g_j(x)$ are continuous real-valued functions on U.

Since $J(C^1, \mathbb{R}^d, a)$ is a finite-dimensional vector space, it has a unique norm, up to bounded equivalence. Hence the C^{1*} norm of the derivation $\langle u, \nabla \cdot (a)\rangle$ is boundedly equivalent to $|u|$. In fact, much more is true. By considering a suitable sequence of C^1 functions one can show that

$$\|\langle u, \nabla \cdot (a)\rangle\|_{C^{1*}} = |u|$$

whenever $a \in \mathbb{R}^d$ and $u \in \mathbb{R}^d$.

Now fix a closed set $E \subset \mathbb{R}^d$ and a point $a \in E$. Each bounded point derivation on $C^1(E)$ at the point a induces a unique bounded point derivation on C^1 at a. Thus, if $d \in J(C^1, E, a)$, then there exists a unique $u \in \mathbb{R}^d$ such that $df = \langle u, \nabla f^*(a)\rangle$ whenever $f \in C^1(E), f^* \in C^1$, and $f = f^* \mid E$. Obviously the map $d \to u$ is one-to-one. In the other direction, if $u \in \mathbb{R}^d$, and $\langle u, \nabla f_1^*(a)\rangle = \langle u, \nabla f_2^*(a)\rangle$ whenever f_1^*, $f_2^* \in C^1$ and $f_1^* = f_2^*$ on E, then it follows that

$$|\langle u, \nabla f^*(a)\rangle| \leqslant |u| \, \|f^*\|_{C^1(E)},$$

hence the formula $df = \langle u, \nabla f^*(a)\rangle$ $(f^* \mid E = f)$ defines a bounded point derivation on $C^1(E)$ at a. Thus, $J(C^1, E, a)$ consists precisely of those functionals

$$f \to \langle u, \nabla f^*(a)\rangle \qquad (f^* \in C^1, f^* \mid E = f)$$

which are well-defined on $C^1(E)$.

We denote the value at the function $f \in C^1(E)$ of the functional corresponding to u by $D_u f(a)$. Thus $D_u f^*(a) = \langle u, \nabla f^*(a)\rangle$ whenever $f^* \in C^1$.

The vectorspace $J(C^1, E, a)$ has dimension between 0 and d. All integral values from 0 to d may occur. For instance, if E is an m-dimensional smooth submanifold of $\mathbb{R}^d$, then $\dim J(C^1, E, a) = m$ for each $a \in E$.

The norm of a derivation $D_u \cdot (a) \in J(C^1, E, a)$ considered as an element of $C^1(E)^*$, is exactly $|u|$. To see this, note first that the inequality

$$|D_u f(a)| \leqslant |u| \, \|f\|_{C^1(E)},$$

is obvious. Conversely, given $\varepsilon > 0$, there exists $f \in C^1$ such that

$$\langle u, \nabla f(a) \rangle \geqslant (|u| - \varepsilon) \, \|f\|_{C^1},$$

hence

$$|D_u f(a)| \geqslant (|u| - \varepsilon) \, \|f\|_{C^1(E)}.$$

Thus

$$\|D_u \cdot (a)\|_{C^1(E)^*} = |u|.$$

The sheaf $J(C^1, E)$ is homeomorphic to a closed subsheaf of the product sheaf $E \times \mathbb{R}^d$. The sets

$$E_j = \{a \in E : \dim J(C^1, E, a) \geqslant j\}$$

are closed subsets of E. For more information on the structure of the sets E_j consult [3, (3.3)] (where the approach is quite different).

Now consider $D^1(E)$. Each bounded point derivation d on $D^1(E)$ at a point $a \in E$ restricts to a bounded point derivation $D_u \cdot (a)$ on $C^1(E)$, since the inclusion map $C^1(E) \to D^1(E)$ is a continuous algebra homomorphism (Lemma 1.1). Moreover, $D_u \cdot (a)$ determines d uniquely, since $C^1(E)$ is dense in $D^1(E)$. We denote df by $d_u f(a)$ for $f \in D^1(E)$. Thus

$$d_u f(a) = D_u f(a) = \langle u, \nabla f(a) \rangle$$

for all $f \in C^1$.

It is tempting to identify $J(D^1, E, a)$ with the space of those $u \in \mathbb{R}^d$ such that $D_u \cdot (a)$ extends continuously from $C^1(E)$ to $D^1(E)$, and in this way to identify $J(D^1, E)$ with a subset of $E \times \mathbb{R}^d$. This procedure is fraught with peril, however, because in general $J(D^1, E)$ is not homeomorphic with its image. This problem will become clearer as we go along.

Since the map $C^1(E) \to D^1(E)$ is a contraction, it follows that $|u| \leqslant \|d_u \cdot (a)\|_{D^1(E)^*}$. It may happen, however, (cf. Section 5) that $\|d_u \cdot (a)\|$ is much larger than $|u|$. By the same token, the sets

$$E'_j = \{a \in E : \dim J(D^1, E, a) \geqslant j\}$$

need not be closed, in general, (unless $d = 1$ or $j = 1$).

The space of bounded point derivations on Lip $(1, E)$ is treated in Sherbert's paper [9, especially §9, pp 264–271]. Sherbert provides three different characterizations of the space $J(\text{Lip}\,(1, \cdot), E, a)$, and other information. Most of his results have analogues for the regular subalgebra $D^1(E)$. There are substantial simplifications, mainly due to the fact that $J(D^1, E, a)$ is finite-dimensional (the dimension of $J(\text{Lip}\,(1, \cdot), E, a)$ is 2^c whenever a is an accumulation point of E). For our purposes, we require the analogue of one of Sherbert's characterisations (cf. Lemma 2.5). We propose to obtain it as a corollary of Sherbert's result. In order to do this, we first establish that every bounded point derivation on $D^1(E)$ extends to a bounded point derivation on Lip (1, E). This in turn reduces to a couple of technical lemmas. The most important asserts that a certain sum of subalgebras of Lip (1, E) is closed.

Fix $a \in E$. Let

$$M = \{f \in \text{Lip}(1, E) : f(a) = 0\},$$
$$J = \{f \in M : f = 0 \text{ on a neighbourhood of } a\},$$
$$\tilde{M} = M \cap D^1(E),$$
$$\tilde{J} = J \cap D^1(E).$$

Then M and J are ideals in Lip (1, E) whereas $\tilde{M}$ and $\tilde{J}$ are ideals in $D^1(E)$. For any space $I \subset \text{Lip}(1, E)$, we use the notation I^2 for the space of finite sums of products of pairs of elements of I. Thus M^2 and $\tilde{M}^2$ are ideals in Lip (1, E) and $D^1(E)$, respectively. By a standard result on Banach algebras [9, (8.4), p. 262], $J(D^1, E, a)$ consists precisely of those functionals in $D^1(E)^*$ which annihilate the constants and $\tilde{M}^2$. Symbolically, we write $J(D^1, E, a) = (\mathbb{R}1 + \tilde{M}^2)^\perp$. Similarly, $J(\text{Lip}(1, \cdot), E, a) = (\mathbb{R}1 + M^2)^\perp$. Sherbert showed [9, (5.2), p. 253] that clos M^2 = clos J. We now establish the same result for $D^1(E)$.

Lemma 2.1. clos $\tilde{M}^2$ = clos $\tilde{J}$.

PROOF. Let $f, g \in \tilde{M}$. Then there exists $f_n, g_n \in C^1$ such that $f_n(a) = g_n(a) = 0$, $\|f - f_n\|_{1,E} \to 0$ and $\|g - g_n\|_{1,E} \to 0$. Thus $\nabla(f_n g_n)(a) = 0$, so that there exists $h_n \in C^1$ such that $h_n = 0$ on a neighbourhood of a, and $\|f_n g_n - h_n\|_C < 1/n$. Thus $h_n \in \tilde{J}$ and $\|h_n - fg\|_{1,E} \to 0$. The result follows.

Thus $(\mathbb{R}1 + M^2)^\perp = (\mathbb{R}1 + J)^\perp$ and $(\mathbb{R}1 + \tilde{M}^2)^\perp = (\mathbb{R}1 + \tilde{J})^\perp$. Hence, to prove that every bounded point derivation on $D^1(E)$ at the point a extends to a bounded point derivation on Lip (1, E) at a, amounts to proving that every annihilator of $\mathbb{R}1 + \tilde{J}$ in $D^1(E)^*$ extends to an annihilator of $\mathbb{R}1 + J$ in Lip $(1, E)^*$. This brings us to the two technical lemmas.

Lemma 2.2. *The vectorspace sum* $D^1(E)$ + clos J *is a closed subalgebra of* Lip (1, E.)

PROOF. The sum of a subalgebra and an ideal is always a subalgebra, so it remains to prove that the sum is closed. For this it suffices to prove that $D^1(E)$ + clos J equals the closure of $C^1 + J$. Obviously, $D^1(E)$ + clos $J \subset$ clos $(C^1 + J)$.

For the converse, fix $f \in$ clos $(C^1 + J)$. Choose a sequence $f_n \in C^1 + J$ such that $\|f - f_n\|_{1,E} < 1/2^{n+1}$. Then $\|f_n - f_{n+1}\|_{1,E} < 1/2^n$. Choose numbers $r_n \swarrow 0$ such that $2r_{n+1} < r_n$ and f_n coincides with a C^1 function on a neighbourhood of the closed ball B_n with centre a and radius r_n. Choose C^1 functions ϕ_n such that $0 \leqslant \phi_n \leqslant 1$, $\phi_n = 1$ on B_{n+1}, $\phi_n = 0$ off B_n, and $|\nabla \phi_n| \leqslant 4/r_n$ (the function $\phi_n(x)$ may be chosen to depend only on $|x - a|$).

Write $f_1 = g_1 + h_1$ with $g_1 \in C^1$ and $h_1 \in J$. Inductively, write $f_{n+1} = g_{n+1} + h_{n+1}$, where $g_{n+1} \in C^1$ and $h_{n+1} \in J$ are chosen as follows. First, define $g_{n+1} = \phi_{n+1} f_{n+1} + (1 - \phi_{n+1}) g_n$. Then $g_{n+1} \in C^1$ as required. Also, $g_{n+1} = f_{n+1}$ on a neighbourhood of a. Define $h_{n+1} = f_{n+1} - g_{n+1}$.

We have

$$g_{n+1} - g_n = \phi_{n+1}(f_{n+1} - f_n).$$

Let $E_n = E \cap B_n$. Then we have the following estimates:

$$\begin{aligned}
\|g_{n+1} - g_n\|_{0,E} &\leqslant \|f_{n+1} - f_n\|_{0,E} \\
\|g_{n+1} - g_n\|'_{1,E} &\leqslant \|\phi_{n+1}\|_{0,E} \|f_{n+1} - f_n\|'_{1,E} + \|\phi_{n+1}\|'_{1,E} \|f_{n+1} - f_n\|_{0,E_{n+1}} \\
&\leqslant \|f_{n+1} - f_n\|'_{1,E} + \frac{4}{r_{n+1}} \cdot r_{n+1} \|f_{n+1} - f_n\|'_{1,E} \\
&= 5 \|f_{n+1} - f_n\|'_{1,E} \\
\|g_{n+1} - g_n\|_{1,E} &\leqslant 5 \|f_{n+1} - f_n\|_{1,E} \\
&\leqslant 5/2^n.
\end{aligned}$$

Thus $\{g_n\}$ is a Cauchy sequence in Lip $(1, E)$, and hence converges to a function $g \in D^1(E)$. Furthermore, $h_n = f_n - g_n \to f - g$, hence $f - g \in \text{clos } J$. Thus $f \in D^1(E) + \text{clos } J$. This shows that $\text{clos }(C^1 + J) \subset D^1(E) + \text{clos } J$, and the result follows.

Lemma 2.3. $\text{clos } \tilde{J} = D^1(E) \cap \text{clos } J$.

PROOF. Clearly, $\text{clos } \tilde{J} = \text{clos }(D^1(E) \cap J) \subset D^1(E) \cap \text{clos } J$. To prove the reverse inequality, fix $f \in D^1(E) \cap \text{clos } J$. Let $\varepsilon > 0$ be given. Choose $g \in C^1$ and $h \in J$ such that $g(a) = 0$, $\|f - g\|_{1,E} < \varepsilon$, and $\|f - h\|_{1,E} < \varepsilon$. Choose $r > 0$ such that $h(x) = 0$ on a neighbourhood of $\{x : |x - a| \leqslant 2r\}$. Choose $\phi \in C^1$ such that $\phi \geqslant 0$, $\phi = 1$ for $|x - a| \leqslant r$, $\phi = 0$ for $|x - a| \geqslant 2r$, and $|\nabla \phi| \leqslant 2/r$. Define $k = (1 - \phi)g$. Then $k \in \tilde{J}$, and $k - g = -\phi g = \phi(h - g)$. As in the previous lemma, we estimate $\|k - g\|_{1,E} \leqslant 5 \|h - g\|_{1,E}$, hence $\|k - f\|_{1,E} \leqslant 11\varepsilon$. Thus $f \in \text{clos } \tilde{J}$.

We can now prove the main lemma.

Lemma 2.4. *Each bounded point derivation on $D^1(E)$ at the point a extends to a bounded point derivation on* Lip $(1, E)$ *at a.*

PROOF. Let $d \in J(D^1, E, a)$. Then d annihilates $\text{clos } \tilde{J}$. Thus, by Lemma 2.3, the formula:

$$L(g + h) = dg (g \in D^1(E), h \in \text{clos } J)$$

determines a well-defined linear functional on $D^1(E) + \text{clos } J$. Let $D^1(E) \oplus \text{clos } J$ denote the outer direct sum of the Banach spaces $D^1(E)$ and $\text{clos } J$, with the norm

$$\|g \oplus h\| = \|g\|_{1,E} + \|h\|_{1,E}.$$

The linear function $\lambda : D^1(E) \oplus \text{clos } J \to D^1(E) + \text{clos } J$, $(g, h) \to g + h$, is continuous and surjective. By Lemma 2.2 and the open mapping theorem, there exists $\kappa > 0$ such that $\inf \{\|g \oplus h\| : g \in D^1(E), h \in \text{clos } J, g + h = f\} \leqslant \kappa \|f\|_{1,E}$ whenever $f \in D^1(E) + \text{clos } J$. It follows easily that L is continuous on $D^1(E) + \text{clos } J$. By the Hahn-Banach theorem, L has a continuous extension d' to Lip (1, E). Since d' annihilates $\mathbb{R}1 + J$, it is a bounded point derivation on Lip (1, E) at the point a.

Armed with Lemma 2.4 we now proceed to transfer Sherbert's results from Lip (1, E) to $D^1(E)$.

We define

$$L(b, c)f = \frac{f(b) - f(c)}{|b - c|}$$

whenever $b, c \in E$ and $f \in$ Lip (1, E). Then $L(b, c)$ is a continuous linear functional on Lip (1, E). It is easily seen that $L(b, c)$ has norm at most 1 and at least $\{1 + |b - c|/2\}^{-1}$ in Lip (1, E)*. Let ψ_a denote the set of all weak-star cluster points in Lip (1, E)* of sequences $\{L(b_n, c_n)\}^{\infty}{}_1$, where $b_n \in E$, $c_n \in E$, $b_n \to a$, and $c_n \to a$. Let $\tilde{\psi}_a$ denote the set of restrictions to $D^1(E)$ of functionals in ψ_a. Let $\tilde{L}(b, c)$ denote the restriction of $L(b\ \ c)$.

Clearly each element of ψ_a is a bounded point derivation on Lip (1 E) at a and each element of ψ_a is a bounded point derivation on $D^1(E)$ at a. We note that ψ_a is a subset of the unit ball in Lip (1 E)* by the Krien-Smulian theorem.

Sherbert [9, (9.3), p. 265] proved that the space J (Lip (1 ·) E a) is the weak-star closure of the linear span of ψ_a in Lip (1 E)*. The result for $J(D^1$ E $a)$ is simpler.

Lemma 2.5. *$J(D^1$ E $a)$ is the linear span of $\tilde{\psi}_a$.*

PROOF. By Lemma 2.4 and the result of Sherbert just quoted, $J(D^1, E, a)$ is the weak-star closure of the linear span of $\tilde{\psi}_a$ in $D^1(E)^*$. But $J(D^1, E, a)$ is finite-dimensional, hence each linear subspace is closed, so the result follows.

Let $\mathscr{P}$ denote the linear span of the point evaluations at points of E. Then Lemma 2.5 implies that $J(D^1, E, a)$ is contained in the weak-star closure of $\mathscr{P}$ in $D^1(E)^*$. Much more than this is true.

Lemma 2.6. *Let $d \in \tilde{\psi}_a$. Then d is the weak-star limit in $D^1(E)^*$ of a sequence $\{\tilde{L}(b_n, c_n)\}$, with $b_n \in E$, $c_n \in E$, $b_n \to a$, and $c_n \to a$.*

PROOF. There exists $u \in \mathbb{R}^d$ such that $d = d_u \cdot (a)$. There exists $d' \in \psi_a$ such that $d' \mid D^1(E) = d$. There exists a sequence $\{L(b'_n, c'_n)\}$ such that $b'_n \in E$, $c'_n \in E$, $b'_n \to a$ $c'_n \to a$, and d' belongs to the weak-star closure of $\{L(b'_n, c'_n)\}$ in Lip (1, E)*.

Thus d belongs to the weak-star closure in $D^1(E)^*$ of $\{\tilde{L}(b'_n, c'_n)\}$. Choose a function $t \in C^1$ such that $t(x) = \langle x, u\rangle$ for x near a. We have $dt = |u|^2$, hence

$$|u|^2 \in \text{clos}\,\{\tilde{L}(b'_n, c'_n)t\} = \text{clos}\left\{\frac{\langle b'_n - c'_n, u\rangle}{|b'_n - c'_n|}\right\}.$$

Thus $|u| = 1$, and we may choose integers $k_n \uparrow \infty$ such that

$$\frac{b_n - c_n}{|b_n - c_n|} \to u$$

where $b_n = b'_{kn}$ and $c_n = c'_{kn}$. Then $\tilde{L}(b_n, c_n)g \to \langle u, \nabla g(a)\rangle$ for all $g \in C^1$.

Fix $f \in D^1(E)$. Let $\varepsilon > 0$ be given. Choose $g \in C^1$ such that $\|f - g\|_{1,E} < \varepsilon$. Choose N such that

$$|\tilde{L}(b_n, c_n)g - \langle u, \nabla g(a)\rangle| < \varepsilon$$

whenever $n > N$. Then

$$\begin{aligned}|\tilde{L}(b_n, c_n)f - d_u f(a)| &\leqslant |\tilde{L}(b_n, c_n)(f - g)| + |\tilde{L}(b_n, c_n)g - d_u g(a)| + |d_u g(a) - d_u f(a)| \\ &\leqslant \|f - g\|_{1,E} + \varepsilon + \|f - g\|_{1,E} \\ &< 3\varepsilon\end{aligned}$$

whenever $n > N$. Thus $\tilde{L}(b_n, c_n)f \to d_u f(a)$. Thus the sequence $\{\tilde{L}(b_n, c_n)\}$ converges to d in the weak-star topology of $D^1(E)^*$. This proves the result.

REMARK. Lemmas 2.5 and 2.6 show that $J(D^1, E, a)$ is contained in the sequential weak-star closure of $\mathscr{P}$ in $D^1(E)^*$. This contrasts with the behaviour of $C^1(E)$. In [8] we showed that $J(C^1, E, a)$ is contained in the sequential weak-star closure of the sequential weak-star closure of $\mathscr{P}$ in $C^1(E)^*$. We use the notation $J_c(C^1, E, a)$ for the space of *sequentially-calculable* bounded point derivations on $C^1(E)$ at a, that is the space of derivations $d \in J(C^1, E, a)$ which are weak-star limits of sequences from $\mathscr{P}$.

In view of the proof of Lemma 2.6, the set $\{u \in \mathbb{R}^d : d_u \cdot (a) \in \tilde{\psi}_a\}$ may be described as the set of all unit vectors $u = \lim\,(b_n - c_n)/|b_n - c_n|$, where $b_n \in E$, $c_n \in E$, $b_n \to a$, and $c_n \to a$. We denote this set by **Tan** (E, a) (cf. [5]). The cone generated by **Tan** $(E\ a)$ has been called the *Denjoy tangent space* of E at the point a. This cone is larger than the cone denoted by **Tan** (E, a) in 3, §(3.3). We denote the set $\{(a, u) : a \in E, u \in \textbf{Tan}\,(E, a)\}$ by **Tan** E, and we use the notation span **Tan** E for the fibre-wise linear span

$$\{(a, u) : a \in E, u \in \text{span}\,\textbf{Tan}\,(E, a)\}.$$

With this notation Lemmas 2.5 and 2.6 yield the following explicit description of $J(D^1, E, a)$.

Theorem 2.7. *Let E be a closed subset of $\mathbb{R}^d$, and let $a \in E$. Then*

(1) $J(D^1, E, a) = \{d_u \cdot (a) : u \in \text{span}\,\textbf{Tan}\,(E, a)\}$;

(2) $J(D^1, E) = \{(a, d_u \cdot (a)) : (a, u) \in \text{span}\,\textbf{Tan}\,E\}$.

3. An extension theorem

In [8] we gave an improved version of the C^1 case of Whitney's extension theorem. The proof of our result was better than the statement, in a manner of speaking. In the present section we formulate a statement which takes full advantage of the proof. This stronger statement is needed for the main theorem in §4. As a corollary, we obtain an explicit description of $J(C^1, E)$, in terms of **Tan** E.

The following lemma is easy to prove.

Lemma 3.1. *Let E be a closed subset of $\mathbb{R}^d$. Then*

(1) **Tan** *E is a closed subset of $E \times \mathbb{R}^d$;*

(2) *For each point $a \in E$, the set* **Tan** *(E, a) is a closed subset of the unit sphere in $\mathbb{R}^d$, symmetric under reflection in the origin.*

Let f be a real-valued function defined on a closed set $E \subset \mathbb{R}^d$. Let $(a, u) \in$ **Tan** E. We say that *$d_u f(a)$ exists* if there exists a number α such that

$$(f(b_n) - f(c_n))/|b_n - c_n| \to \alpha$$

whenever $b_n \in E$, $c_n \in E$, $b_n \to a$, $c_n \to a$, and $(b_n - c_n)/|b_n - c_n| \to u$. If this is so, then we say that $d_u f(a)$ equals α. Obviously, if $f \in D^1(E)$, then $d_u f$ (a) exists for each $(a, u) \in$ **Tan** E, and equals $d_u f(a)$ (as defined in §2). So the definition just given extends $d_u \mid (a)$ from $D^1(E)$ to a certain domain contained in $\mathbb{R}^E$.

Theorem 3.2. *Let E be a closed subset of $\mathbb{R}^d$. Let f be a real-valued function on E. Suppose the following three conditions are satisfied:*

(1) *$d_u f(a)$ exists for all $(a, u) \in$* **Tan** *E;*

(2) *for each $a \in E$, $d_u f(a)$ has a linear extension to* span **Tan** (E, a);

(3) *$d_u f(a)$ has a continuous extension from span* **Tan** *E to* clos span **Tan** E.

Then f has a C^1 extension to $\mathbb{R}^d$.

As we remarked above, the proof in [8] actually proves the above result. Some slight adjustments are needed, but it is not worth giving the details.

REMARK. Not only is Theorem 3.2 more explicit than the result of [8], it is also completely elementary. The statement and proof require only advanced calculus.

Corollary 3.3. *Let E be a closed subset of $\mathbb{R}^d$.*
Then $J(C^1, E) = \{(a, D_u \cdot (a)) : (a, u) \in$ clos span **Tan** $E\}$.

PROOF. See [8, Corollary, p. 320].

Problem. Is it always the case that each derivation in $J_c(C^1, E, a)$ extends continuously to $D^1(E)$?

4. Main Theorem

In this section we characterise the closed sets E such that $D^1(E) = C^1(E)$. The result is as follows.

Theorem 4.1. *Let E be a closed subset of $\mathbb{R}^d$. Then the following three conditions are equivalent:*

(1) *$D^1(E) = C^1(E)$, that is every function on E which is a* Lip $(1, E)$ *limit of C^1 functions has a C^1 extensions;*

(2) *there exists $\kappa < 0$ such that*

$$\|f\|_{C^1(E)} \leqq \kappa \|f\|_{1,E}$$

whenever $f \in C^1(E)$;

(3) *there exists $\kappa > 0$ such that every bounded point derivation d on $C^1(E)$ at a point of E extends to a bounded point derivation d' on $D^1(E)$ with*

$$\| d' \|_{D^1(E)^*} \leqq \kappa \|d\|_{C^1(E)^*}.$$

REMARK 1. The main result of this paper is the equivalence of (1) and (3).

REMARK 2. A sufficient (but not necessary) condition for (2) is that E be *uniformly regular,* in the sense of Dales and Davie [2, p. 30]. Incidentally, the space $D^1(E)$ of [2] is *not* the complexification of ours. Their $D^1(E)$ is defined for $E \subset \mathbb{C} \approx \mathbb{R}^2$ and consists of functions with a *complex* derivative at all points of E.

REMARK 3. A sufficient (but not necessary) condition for (3) is that E be *uniformly 1-thick,* in the sense of [7, p. 381]. A weaker condition is the following:

There exists $\kappa > 0$ such that for every $a \in E$ there exists a basis $\{u_1, \ldots, u_{p_a}\}$ for span **Tan** (E, a) *such that $u_j \in$* **Tan** (E, a), *$|u_j| = 1$ and*

$$\left|u_j - \sum_k \alpha_k u_k\right| \geqslant \kappa$$

whenever $x_k \in \mathbb{R}$ and $\alpha_j = 0$.

REMARK 4. We emphasise that (1) asserts the identity of $D^1(E)$ and $C^1(E)$ *as sets,* not as Banach spaces. We conjecture that $D^1(E)$ and $C^1(E)$ are isometric if and only if $J(D^1, E, a)$ and $J(C^1, E, a)$ are isometric for all $a \in E$.

REMARK 5. We construct an example E in §5 such that $J(D^1, E) \neq J(C^1, E)$. We construct another example in which $J(D^1, E) = J(C^1, E)$, but the norms are not equivalent. This shows that the norm condition cannot be omitted in (3).

Before proving (4.1) we need a lemma.

Lemma 4.2. *Let $f \in D^1(E)$ and let $a \in E$. Then $d_u f(a)$ is linear in u on* span **Tan** (E, a).

PROOF. If $f \in C^1(E)$, then the result is obvious. Hence, by continuity, it holds for $f \in D^1(E)$.

PROOF OF 4.1. By the open mapping theorem, (1) implies (2). Obviously, (2) implies (3). So it remains to prove that (3) implies (1).

Suppose (3) holds. Fix $f \in D^1(E)$. We wish to prove that f has a C^1 extension. By Lemma 4.2 and Theorem 3.2, we need only show that $d_u f(a)$ has a continuous extension from span **Tan** E to clos span **Tan** E. By the hypothesis (3), Theorem 2.7 and Corollary 3.3, span **Tan** E is closed. Thus we need only show that $d_u f(a)$ is continuous on span **Tan** E.

Fix $(a, u) \in$ span **Tan** E. Let $\varepsilon > 0$ be given. Choose $g \in C^1$ such that

$$\|f - g\|_{1,E} < \varepsilon/3\kappa(|u| + 1).$$

Choose $\delta > 0$ such that

$$|\langle u, \nabla g(a)\rangle - \langle v, \nabla g(b)\rangle| < \varepsilon/3$$

whenever $|a - b| + |u - v| < \delta$. Then

$$\begin{aligned} |d_u f(a) - d_v f(b)| &\leqslant |d_u f(a) - d_u g(a)| + |\langle u, \nabla g(a)\rangle - \langle v, \nabla g(b)\rangle| + |d_v g(b) - d_v f(b)| \\ &\leqslant \|d_u \cdot (a)\|_{D^1(E)^*} \|f - g\|_{1,E} + \varepsilon/3 + \|d_v \cdot (b)\|_{D^1(E)^*} \|f - g\|_{1,E} \\ &\leqslant (\kappa|u| + \kappa|v|)\,\|f - g\|_{1,E} + \varepsilon/3 \\ &< \varepsilon, \end{aligned}$$

whenever $(b, v) \in$ span **Tan** E and $|a - b| + |u - v| < \delta$. This proves that $d_u f(a)$ is continuous on span **Tan** E, and the result follows.

Corollary 4.3. *Let $E \subset \mathbb{R}$ be closed. Then $D^1(E) = C^1(E)$.*

PROOF. At each isolated point a of E, we have **Tan** $(E, a) = \phi$, hence $J(D^1, E, a) = J(C^1, E, a) = \{0\}$. At each accumulation point a of E we have **Tan** $(E, a) = \{1, -1\}$, so that $J(D^1, E, a) = J(C^1, E, a) \cong \mathbb{R}$, and the two spaces of derivations are isometric. Thus condition (3) is satisfied.

REMARK. It is not difficult to find a direct constructive proof of Corollary 3.2, and to show that $D^1(E)$ is actually isometric to $C^1(E)$ when $E \subset \mathbb{R}$.

5. Three examples

Example 5.1. Fix $R > 1$ and $0 < \theta < \pi$. Choose positive numbers r_n, s_n such that $r_1 = 1/R$, $r_{n+1} < r_n$, $s_{n+1} < s_n$, $Rr_{n+1} < s_n$, $Rs_n < r_n$, $s_n/r_n \to 0$, $r_{n+1}/s_n \to 0$. Define $a_n = (r_n, 0)$, $b_n = (s_n \cos\theta, s_n \sin\theta)$. Let $E(\theta, R)$ denote the compact set

$$\{0, a_1, b_1, a_2, b_2, \ldots,\}.$$

Clearly, **Tan** $E = \{0\} \times$ **Tan** $(E, 0)$ and **Tan** $(E, 0) = \{\pm(1, 0), \pm(\cos\theta, \sin\theta)\}$. Consider the function $f(x, y) = y/\sin\theta$ $((x, y) \in E)$. Clearly, f has a C^1 extension, and

$$\|f\|_{1,E} \leqslant \frac{1}{R^2} + \frac{R}{R-1} \leqslant \frac{R+1}{R-1},$$

$$d_{(0,1)} f(0) = \frac{1}{\sin\theta}.$$

Thus

$$\|d_{(0,1)} \cdot (0)\|_{D^1(E)^*} \geqslant \frac{R-1}{R+1} \cdot \frac{1}{\sin\theta} \geqslant \frac{1}{2\sin\theta}$$

provided R is greater than 3. This shows that

$$\|d_u \cdot (a)\| / |u|$$

can be arbitrarily large.

In this example, **Tan** $(E, 0)$ has four points. By an obvious modification of the construction one can obtain a compact set $E \subset \mathbb{R}^d$ with **Tan** $(E, 0)$ equal to any preassigned symmetric subset of the unit sphere in $\mathbb{R}^d$.

Example 5.2. Consider numbers $\theta_n \downarrow 0$ and $R_n \uparrow \infty$ such that $R_n > 3$ and

$$R_n > 2 \max \{|\theta_n - \theta_{n-1}|^{-1}, |\theta_n - \theta_{n+1}|^{-1}\}.$$

Let

$$E_n = \{(\theta_n, 0) + (x, y) : (x, y) \in E(\theta_n, R_n)\},$$

$$E = \{0\} \bigcup_{n=1}^{\infty} E_n .$$

Then E is compact. Provided the R_n are large enough, we have

$$\|d_u \cdot (a_n)\|_{D^1(E)} \geqslant \frac{1}{2\sin\theta_n} \uparrow \infty,$$

where $u = (0, 1)$ and $a_n = (\theta_n, 0)$. Thus condition (3) of Theorem 4.1 is violated, and so $D^1(E) \neq C^1(E)$.

Furthermore, provided $R_n\theta_n \uparrow \infty$, we have **Tan** $(E, 0) = \{\pm (1, 0)\}$, so that $J(D^1, E, 0)$ has dimension 1. But $J(C^1, E, a_n)$ has dimension 2 for each n, hence $0 = \lim a_n \in \text{clos } E_1 = E_1$, that is $J(C^1, E, 0)$ has dimension 2.

Example 5.3. Let E be the set of the last example, and form

$$H = E \cup \{(0, y) : 0 < y < 1\}.$$

Then $J(D^1, H, a) = J(C^1, H, a)$ for every point $a \in H$, but $\|d_u \cdot (a_n)\|_{D^1(H)^*} \uparrow \infty$ with $u = (0, 1)$ and $a_n = (\theta_n, 0)$. Thus every bounded point derivation on $C^1(E)$ extends to $D^1(E)$, but the norm condition fails.

6. A sufficient condition for $f \in D^1(E)$

Here we present another application of the results in Section 2.

We begin by proving that for $f \in D^1(E)$, the function $d_u f(a)$ is continuous on **Tan** E, as a function of (a, u). This allows us to ask the following question: suppose $f \in \text{Lip}(1, E)$, and $d_u f(a)$ is continuous for $(a, u) \in$ **Tan** E. What additional hypothesis will guarantee that $f \in D^1(E)$? For each point $a \in E$, the set **Tan** (E, a) consists of a number of directions. We show that if f has an extension in Lip $(1, \mathbb{R}^d)$ such that the directional derivative $\langle u, \nabla f(a)\rangle$ (defined a.e.) is approximately-continuous at a whenever $u \in$ **Tan** (E, a), then $f \in D^1(E)$.

Lemma 6.1. *Let $f \in D^1(E)$. Then $d_u f(a)$ is continuous on* **Tan** *E, as a function of (a, u).*

PROOF. Fix $(a, u) \in$ **Tan** E, and let $\varepsilon > 0$ be given. Choose $g \in C^1$ such that $\|f-g\|_{1,E} < \varepsilon$. Choose $\delta > 0$ such that

$$|\langle u, \nabla g(a)\rangle - \langle v, \nabla g(b)| < \varepsilon$$

if $|a-b| + |u-v| < \delta$. Then

$$\begin{aligned}|d_u f(a) - d_v f(b)| &\leqslant |d_u f(a) - d_u g(a)| + |\langle u, \nabla g(a)\rangle - \langle v, \nabla g(b)| + |dg(b_v) - d_v f(b)| \\ &< \|f-g\|_{1,E} + \varepsilon + \|f-g\|_{1,E} \\ &< 3\varepsilon\end{aligned}$$

whenever $(b, v) \in$ **Tan** E and $|a-b| + |u-v| < \delta$, since $d_u \cdot (a)$ and $d_v \cdot (b)$ have norm 1 in $D^1(E)^*$. Thus $d_u f(a)$ is continuous on **Tan** E.

Let $\mathscr{L}^d$ denote Lebesque measure on $\mathbb{R}^d$. If f belongs to Lip $(1, \mathbb{R}^d)$, then the Frechet derivative $\nabla f(a)$ exists $\mathscr{L}^d$ almost everywhere [3, (3.1.6), p. 216]. For each $u \in \mathbb{R}^d$, the directional derivative $\langle u, \nabla f(a)\rangle$ is a measurable function of a, and belongs to $L^\infty(\mathbb{R}^d, \mathscr{L}^d)$. Indeed,

$$\|\langle u, \nabla f(\cdot)\|_{L^\infty} \leqslant |u| \, \|f\|_{1,\mathbb{R}^d}$$

for all $u \in \mathbb{R}^d$.

It is important to distinguish between $d_u f(a)$ and $\langle u, \nabla f(a)\rangle$. If $f \in$ Lip $(1, \mathbb{R}^d)$, then $d_u f(a)$ may exist for some (or none, or all) $(a, u) \in$ span **Tan** E. It depends only on the restriction of f to E. On the other hand, $\langle u, \nabla f(a)\rangle$ exists for $\mathscr{L}^d$ almost all a and all u. If $f \in C^1$, and $(a, u) \in$ span **Tan** E, then the two exist and are equal.

The existence of $d_u f(a)$ neither implies nor follows from the existence of $\langle u, \nabla f(a)\rangle$. Furthermore, even if both exist, they need not be equal.

Recall that a real-valued measurable function $h(x)$ is *approximately-continuous* at a point $a \in \mathbb{R}^d$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathscr{L}^d\{x \in \mathbb{R}^d : |x-a| \leqslant t, |h(x) - h(a)| > \varepsilon\} \leqslant t^d \varepsilon$$

whenever $0 < t < \delta$ (that is, the inverse image of each neighbourhood of $h(a)$ has full density at a).

Theorem 6.2. *Let E be a compact subset of $\mathbb{R}^d$, and let $f \in$ Lip $(1, \mathbb{R}^d)$. Suppose that the following two conditions hold:*

(1) *$d_u f(a)$ exists for all $(a, u) \in$* **Tan** *E;*

(2) *for each $(a, u) \in$* **Tan** *E, the function $h(x)$ defined by*

$$h(x) = \begin{cases} \langle u, \nabla f(x)\rangle, & \text{for a.a. } x \neq a, \\ d_u f(a), & \text{for } x = a \end{cases}$$

is approximately-continuous at a. Then $f \mid E \in D^1(E)$.

PROOF. We may assume that $\|f\|_{1,\mathbb{R}^d} = 1$. This implies that $|\langle u, \nabla f(a)\rangle| \leqslant |u|$ almost everywhere, hence, by condition (2), $|d_u f(a)| \leqslant 1$ for $(a, u) \in \mathbf{Tan}\, E$.

Choose a sequence $\{\phi_n\}^\infty{}_1$ of C^1 functions such that

(*a*) $0 \leqslant \phi_n \leqslant (10n)^d$,

(*b*) $\phi_n = 0$ off the ball $B_n = \{x \in \mathbb{R}^d : |x| < 1/n\}$,

(*c*) $\int \phi_n \, d\mathscr{L}^d = 1$.

Define f_n as the convolution $\phi_n * f$. Then f_n is a sequence of C^1 functions. We claim that f_n converges to f weakly in Lip $(1, E)$.

We shall prove this claim by using the following characterisation of sequential weak convergence in Lip (1, E), due to Sherbert [9, (9.9), p. 269]:

Let E be compact, and let $f, f_n \in$ Lip $(1, E)$. Then $f_n \to f$ weakly in Lip $(1, E)$ if and only if the following three conditions hold:

(i) $\{f_n\}$ is bounded in Lip $(1, E)$ norm,

(ii) $f_n \to f$ pointwise on E

(iii) $df_n \to df$ for all bounded point derivations, d, on Lip $(1, E)$.

Fix $(a, u) \in \mathbf{Tan}\, E$. Let $\varepsilon > 0$ be given. Choose N such that

$$\mathscr{L}^d\{x : |x - a| \leqslant \frac{1}{n}, |\langle u, \nabla f(x)\rangle - d_u f(a)| > \varepsilon\} < \frac{\varepsilon}{n^d}$$

whenever $n \geqslant N$. Then for $n \geqslant N$, we have

$$\begin{aligned} |\langle u, \nabla f_n(a)\rangle - d_u f(a)| &= |\langle u, (\phi_n * \nabla f)(a)\rangle - d_u f(a)| \\ &= \left| \int_{B_n} \phi_n(x)\{\langle u, \nabla f(a - x)\rangle - d_u f(a)\}\, d\mathscr{L}^d x \right| \\ &\leqslant \int_{B_n} \phi_n(x)\, |\langle u, \nabla f(a - x)\rangle - d_u f(a)|\, d\mathscr{L}^d x. \end{aligned}$$

Now $|\langle u, \nabla f(a - x)\rangle - d_u f(a)| \leqslant \varepsilon$ for $x \in B_n \sim A$, where $\mathscr{L}^d A < \varepsilon/n^d$, hence we may continue the inequality:

$$\begin{aligned} &\leqslant \int_{B_n \sim A} \phi_n(x)\, \varepsilon\, d\mathscr{L}^d + \int_A (10n)^d \cdot 2 d\mathscr{L}^d \\ &\leqslant (1 + 10^d \cdot 2)\varepsilon. \end{aligned}$$

Thus $d_u f_n(a) = \langle u, \nabla f_n(a)\rangle \to d_u f(a)$ as $n \to \infty$. This holds for all $(a, u) \in \mathbf{Tan}\, E$.

For almost all $a \in \mathbb{R}^d$, the function $\langle u, \nabla f(a)\rangle$ is linear in u. Hence by condition (2), for all $a \in E$, the function $d_u f(a)$ has a linear extension from $\mathbf{Tan}\,(E, a)$ to span $\mathbf{Tan}\,(E, a)$. We denote this extension by the same symbol $d_u f(a)$.

Let d be any bounded point derivation on Lip $(1, E)$. Then the restriction $d \mid D^1(E)$ equals $d_u \cdot (a)$ for some $(a, u) \in$ span **Tan** E. Let $d' \varepsilon$ span ψ_a be such that $d' \mid D^1(E) = d_u \cdot (a)$. Then $d' = \alpha_1 d_1 + \ldots + \alpha_m d_m$, where $\alpha_j \in \mathbb{R}$ and $d_j \in \psi_a$. Let $d_j \mid D^1(E)$ be $d_{u_j} \cdot (a)$. Then $u = \alpha_1 u_1 + \ldots + a_m u_m$, and there exists $b^j{}_n, c^j{}_n \in E$ such that $b^j{}_n \to a$, $c^j{}_n \to a$, $(b^j{}_n - c^j{}_n)/|b^j{}_n - c^j{}_n| \to u_j$, and d_j belongs to the weak-star closure of $\{L(b^j{}_n, c^j{}_n)\}$ in Lip $(1, E)^*$. Since $d_{u_j} f(a)$ exists, we have $L(b^j{}_n, c^j{}_n) f \to d_{u_j} f(a)$, hence $d_j f = d_{u_j} f(a)$. Since $d_u f(a)$ is linear in u, we obtain $d'f = d_u f(a)$. Now d belongs to the closure of the set of $d' \varepsilon$ span ψ_a such that $d' \mid D^1(E)$ equals $d_u \cdot (a)$. Thus $df = d_u f(a)$

Furthermore,

$$\begin{aligned} d_u f(a) &= \sum_j \alpha_j\, d_{u_j} f(a) \\ &= \Sigma\, \alpha_j \lim_n d_{u_j} f_n(a) \\ &= \lim_n d_u f_n(a) \\ &= \lim_n df_n. \end{aligned}$$

Thus $df_n \to df$. So condition (c) holds.

It is easy to check conditions (a) and (b). Thus, f_n converges to f weakly in Lip $(1, E)$, as claimed. So f belongs to the weak closure of C^1 in Lip $(1, E)$. But the weak closure of a linear subspace coincides with its norm closure, hence $f \in D^1(E)$.

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