FIVE GENERALISATIONS OF THE WEIERSTRASS APPROXIMATION THEOREM

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Abstract

This paper explains how the Weierstrass theorem may be generalised in five distinct ways without going beyond two dimensions.

Introduction

The Weierstrass theorem states that each real-valued continuous function on the interval $[0,1]$ is the uniform limit on $[0,1]$ of a sequence of polynomials $p_n(x) \in \mathbb{R}[x]$. It is remarkable in how many ways it can be generalised and extended. My purpose in this paper is to describe five natural generalisations. Two are famous. The others are less well known. Each way involves uniform polynomial approximation in the plane, and they are all quite distinct.

If F is $\mathbb{R}$ or $\mathbb{C}$, and $f_1,\dots,f_n$ are F-valued functions on a compact space X, then $F[f_1,\dots,f_n]$ denotes the F-algebra of functions generated by $f_1,\dots,f_n$. We denote by $C(X,F)$ the uniform algebra of all continuous F-valued functions on X.

When $[0,1]$ is replaced by an arbitrary compact set $X \subset \mathbb{R}^2 = \mathbb{C}$, then there are a number of options for what should replace $p(x) \in \mathbb{R}[x]$. The most obvious are:

(1) $p(x) \in \mathbb{R}[x]$, $p(x,y) \in \mathbb{R}[x,y]$, or $p(f_1,\dots,f_n) \in \mathbb{R}[f_1,\dots,f_n]$, where each f_j belongs to $C(X,\mathbb{R})$.

(2) $p(z) \in \mathbb{C}[z]$, or $p(f) \in \mathbb{C}[f]$, where $f \in C(X,\mathbb{C})$.

(3) $p(f_1,\dots,f_n) \in \mathbb{C}[f_1,\dots,f_n]$, where each $f_j \in C(X,\mathbb{C})$.

(4) $p(x) + q(y) \in \mathbb{R}[x] + \mathbb{R}[y]$, or $p_1(f_1) + \dots + p_n(f_n) \in \mathbb{R}[f_1] + \dots + \mathbb{R}[f_n]$, where each $f_j \in C(X,\mathbb{R})$.

(5) $p(z) + q(\bar{z}) \in \mathbb{C}[z] + \mathbb{C}[\bar{z}]$, or $p_1(f_1) + \dots + p_n(f_n) \in \mathbb{C}[f_1] + \dots + \mathbb{C}[f_n]$, where each $f_j \in C(X,\mathbb{C})$.

In each case we may ask: for which X is the given class of polynomials dense in $C(X,\mathbb{R})$ or $C(X,\mathbb{C})$, whichever is appropriate? Some of these questions are very difficult, and not all the answers are known.

1. Case (1) : Stone's theorem

The complete solution is given by Stone's theorem [2] which in this context takes the following form.

Theorem. *Let $X \subset \mathbb{R}^2$ be compact and let $f_1,...,f_n \varepsilon C(X,\mathbb{R})$. Then a necessary and sufficient condition for $\mathbb{R}[f_1,...,f_n]$ to be dense in $C(X,\mathbb{R})$ is that $f_1,...,f_n$ separate points on X*, (i.e. for each pair $a \neq b$ of points of X, there exist j such that $f_j(a) \neq f_j(b)$).

In particular, $\mathbb{R}[x,y]$ is always dense in $C(X, \mathbb{R})$, and $\mathbb{R}[x]$ is dense in $C(X, \mathbb{R})$ if and only if the projection of X into the x-axis is one-one.

2. Case (2) : Lavrentieff's theorem

2.1 For $p(z)$, the complete solution is given by Lavrentieff's theorem [2]

Theorem. *Let $X \subset \mathbb{C}$ be compact, with no interior. Then $\mathbb{C}[z]$ is dense in $C(X,\mathbb{C})$ if and only if $\mathbb{C} \sim X$ is connected.*

2.2 For $p(f)$, we obtain the following immediate corollary.

Corollary. *Let $X \subset \mathbb{C}$ be compact, with no interior, and let $f \ \varepsilon\ C(X,\mathbb{C})$. Then $\mathbb{C}[f]$ is dense in $C(X,\mathbb{C})$ if and only if f is injective and $\mathbb{C} \sim f(X)$ is connected.*

3. Case (3) : Approximation on surfaces in $\mathbb{C}^n$

3.1. If a real-valued function h belongs to $\mathbb{C}[f_1,...,f_n]$, then the extreme norm 1 annihilating measures on X for $\mathbb{C}[f_1,...,f_n]$ are supported on the level sets of h. This is seen by an argument of de Branges (see [5]). Because of this fact, it is enough to consider the case in which $\mathbb{C}[f_1,...,f_n]$ contains no nonconstant real-valued functions.

3.2. The problem can be rephrased as follows. Let Y denote the set in $\mathbb{C}^n$ defined by

$$Y = \{f_1(w), ..., f_n(w) : w \ \varepsilon\ X\}.$$

Then $\mathbb{C}[f_1,...,f_n]$ is uniformly dense in $C(X, \mathbb{C})$ if and only if $f_1,...,f_n$ separate points on X and $\mathbb{C}[z_1,...,z_n]$ is uniformly dense in $C(Y,\mathbb{C})$.

Now by the Oka-Weil theorem $\mathbb{C}[z_1,...,z_n]$ is dense in $\mathbb{C}(Y,\mathbb{C})$ if and only if (1) Y is polynomially-convex, and (2) $\mathcal{O}\ (Y)$ (= the space of all functions holomorphic on a neighbourhood of Y in $\mathbb{C}^n$) is dense in $C(Y)$. This allows us to break the problem into two parts.

Our understanding of the second part is more satisfactory than that of the first. Let $X = D$, the closed unit disc. Then Y is a bordered real submanifold of $\mathbb{C}^n$. If Y is a bordered complex submanifold, then obviously $\mathcal{O}\ (Y)$ is not dense in $C(Y, \mathbb{C})$, because all functions in the closure of $\mathcal{O}\ (Y)$ are analytic on the interior of Y. This fact makes it reasonable to look for measures of non-analyticity of Y. The most obvious condition, for C^1 functions f_j, is that for each $a \ \varepsilon\ D$ at least one of the products $df_j(a) \wedge df_k(a)$ be nonzero. This is equivalent to saying that Y has no complex tangents, i.e. that the real tangent plane to Y is never a complex line. The following theorem is due in its final form to Range and Siu [8]. Earlier versions, with more stringent smoothness requirements, were proved by Hörmander and Wermer, and Nirenberg and Wells.

Theorem. *Suppose Y is a bordered C_1 surface in $\mathbb{C}^n$, with no complex tangents. Then $\mathcal{O}\ (Y)$ is dense in $C(Y,\mathbb{C})$.*

3.3. Suppose now that $f_j : D \to \mathbb{C}$ are C_1 functions, separate points on D, and that the corresponding surface $Y \subset \mathbb{C}^n$ has no complex tangents. What additional hypothesis will make Y polynomially-convex? Not much is known about this. Results to date concern the case $\mathbb{C}[z,f]$. It is conjectured that if f is a direction-reversing homeomorphism, then

$$Y = \{(z,w) : w = f(z),\ z \in D\}$$

is polynomially-convex. In terms of derivatives, f is locally direction-reversing if and only if $|f_{\bar z}| > |f_z|$, where

$$f_z = \tfrac{1}{2}(f_x - if_y),$$
$$f_{\bar z} = \tfrac{1}{2}(f_x + if_y).$$

We quote some partial results.

Theorem. (i) (Wermer [9]) *Suppose* $f = \bar z + R$, *and there exists* κ, $0 < \kappa < 1$, *such that* $|R(z_1) - R(z_2)| \leqslant \kappa|z_1 - z_2|$ *for all* $z_1, z_2 \in D$. *Then* Y *is polynomially-convex.*

(ii) [6] *Suppose* Re $f_{\bar z} \geqslant |f_z|$ a.e. *in* D. *Then* Y *is polynomially-convex.*

(iii) [7] *Suppose* $\bar f$ *is analytic on* int D. *Then* Y *is polynomially-convex.*

3.4. A noteworthy result for three functions is the following [5].

Theorem. *Suppose* f, g, *and* h *are homeomorphisms of* $\mathbb{C}$ *onto* $\mathbb{C}$, *and* deg $f = -$deg g. *Then* $\mathbb{C}[f, g, |h|]$ *is dense in* $C(D, \mathbb{C})$.

4. Case (4) : Tchebyshev approximation

4.1. Let $X \subset \mathbb{R}^2$ be compact, and let $\pi_j = X \to \mathbb{R}$ $(j = 1,2)$ be the projections of X into the coordinate axes, i.e. the functions x and y.

Consider the question: for which X is $\mathbb{R}[x] + \mathbb{R}[y]$ dense in $C(X, \mathbb{R})$? In view of the Weierstrass theorem, the closure of $\mathbb{R}[x] + \mathbb{R}[y]$ is the same as the closure of the space of functions of the form $f(x) + g(y)$, with continuous f and g.

A *trip* in X is a sequence $a_1, a_2, \ldots$ of points of X such that

$$\pi_1(a_1) = \pi_1(a_2),\ \pi_2(a_2) = \pi_2(a_3),\ \pi_1(a_3) = \pi_1(a_4)\ ,\ldots .$$

Given a trip $t = \{a_n\}$, consider the point masses $\delta(a_n)$ and the measures

$$\mu_n(t) = \frac{1}{n}\{\delta(a_1) - \delta(a_1) + \ldots + (-1)^{n-1}\delta(a_n)\}.$$

These measures μ_n belong to the unit ball of the space $M(X, \mathbb{R}) = C(X, \mathbb{R})^*$ and so they have at least one weak-star limit point (possibly the zero measure). Each weak-star limit point μ is an annihilating measure for $\mathbb{R}[x] + \mathbb{R}[y]$. If the sequence $\{\mu_n\}$ is actually weak-star convergent to μ, then we say that μ is *generated* by the trip t.

Theorem (Marshall and O'Farrell). *Each extreme norm* 1 *annihilating measure for* $\mathbb{R}[x] + \mathbb{R}[y]$ *is generated by some trip.*

The proof of this theorem will appear in a later paper. An immediate corollary is the solution to the approximation problem.

4.2. Corollary. $\mathbb{R}[x] + \mathbb{R}[y]$ *is dense in* $C(X, \mathbb{R})$ *if and only if* $\mu_n(t)$ *tends weak-star to zero for each trip.*

This gives an effective criterion for analysing examples.

4.3. In a special case, a simpler criterion works. The *orbit* of a point $a \in X$ is the set of all points $b \in X$ such that $a = a_1$ and $b = a_m$ for some trip $\{a_n\}$ and some m. A *round trip* is a periodic trip with $a_n \neq a_{n+1}$ for each n.

Theorem [4.7]. *Suppose each orbit is closed. Then* $\mathbb{R}[x] + \mathbb{R}[y]$ *is dense in* $C(X, \mathbb{R})$ *if and only if there are no round trips.*

Another way of putting it is that $\mathbb{R}[x] + \mathbb{R}[y]$ is dense in $C(X, \mathbb{R})$ if and only if $\mathbb{R}[x] + \mathbb{R}[y]$ is dense in $C(Y, \mathbb{R})$ for each *finite* $Y \subset X$ (provided each orbit is closed). There is an example of Havinson which shows that this criterion does not work in general [3].

4.4. These results generalise without difficulty to the vectorspace sum $A_1 + A_2$ of two subalgebras (with identity) of $C(X, \mathbb{R})$. Let x_j denote the quotient space of X obtained by identifying points which A_j fails to separate, and let $\pi_j : X \to X_j$ be the natural projection map. Then *trip, orbit, generated,* and *round trip* may be defined as above, and all the results go through.

The sum of more than two subalgebras presents formidable problems.

4.5. We may also ask when each $h(x,y) \in C(X,\mathbb{R})$ may be *represented* as a sum $f(x) + g(y)$ with continuous f and g. This is solved by combining Corollary 4.2 with the following.

Let Z denote the quotient space of X induced by $\mathbb{R}[x] \cap \mathbb{R}[y]$, and let $\pi : X \to Z$ be the projection. This means that the (closed) sets $E = \pi^{-1}(z)$ are maximal with respect to the property that $f(x) = g(y)$ forces $f =$ constant on E. This means that E contains the closure of the orbit of each point of E.

$$\text{For } T \subset X, \operatorname*{var}_T f \text{ denotes } \sup_T f - \inf_T f.$$

Theorem [4, Prop.4]. *The subspace*

$$\text{clos } \mathbb{R}[x] + \text{clos } \mathbb{R}[y]$$

is closed in $C(X, \mathbb{R})$ *if and only if there exists* $\kappa > 0$ *such that*

$$\sup_{z \in Z} \operatorname*{var}_{\pi^{-1}(z)} f(x) \leqslant \kappa \sup_{y \in \pi_2(X)} \operatorname*{var}_{\pi_2^{-1}(y)} f(x)$$

for all continuous functions $f : \pi_1(X) \to \mathbb{R}$.

A more explicit version of this condition is that there exist an integer N such that, given $z \varepsilon Z$ and $a,b \varepsilon \pi_1\pi^{-1}(z)$, there exists $n < N$ and $a = a_1, a_2, ..., a_{n+1} = b \varepsilon \mathbb{R}$, and $y_1, ..., y_n \varepsilon \pi_2(X)$ such that a_j and a_{j+1} belong to $\pi_1\pi_2^{-1}(y_j)$, for $j = 1, ..., n$.

5. Case (5) : Generalised Walsh-Lebesgue theorem

5.1. $p(z) + q(\bar{z})$ is covered by the following.

Theorem (Walsh-Lebesgue [2]). $\mathbb{C}[z] + \mathbb{C}[\bar{z}]$ *is dense in* $C(X,\mathbb{C})$ *if and only if* X *is the boundary of a compact set* Y *with connected complement* $\mathbb{C} \sim Y$.

5.2. For $\mathbb{C}[f_1] + \mathbb{C}[f_2]$, the main result available concerns homeomorphisms.

Theorem [5]. *Let f and g be homeomorphisms of C onto* $\mathbb{C}$ *with* $\deg f = -\deg g$. *Let* $X = bdy\ Y$, *where* $Y \subset \mathbb{C}$ *is compact and* $\mathbb{C} \sim Y$ *is connected. Then* $\mathbb{C}[f] + \mathbb{C}[g]$ *is dense in* $C(X,\mathbb{C})$.

The Walsh-Lebesgue theorem is the case $f = z$, $g = \bar{z}$. The case $X = S^1$, the unit circle, is due to Browder and Wermer [1].

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