RESTRICTION ALGEBRAS

By A. G. O'FARRELL, M.R.I.A.

Department of Mathematics, Maynooth College

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ABSTRACT

Let A be a uniform algebra on a compact Hausdorff space X. We discuss the family of sets $E \subset X$ for which the restriction algebra $A \mid E$ is uniformly closed. Without loss of generality, it suffices to consider A-convex sets E. For a moderately large family of algebras we characterise the A-convex sets E such that $A \mid E$ is closed.

1. Introduction

(1.1). Let A be a complex uniform algebra on a compact Hausdorff space X [2, p.32]. This paper is about the family of sets $E \subset X$ for which the restriction algebra $A \mid E$ is uniformly closed. For background, see [1, 2, 3, 5, 6, 7, 8].

Let E be a subset of X. The *A-convex hull* of E (in X) is the set

$$\{a \in X : |f(a)| \leqslant \|f\|_E \text{ whenever } f \in A\},$$

where $\|f\|_E$ denotes the uniform norm on E. We say that E is *A-convex* if it equals its A-convex hull. The kernel of E is the ideal

$$\ker E = \{f \in A : f \mid E = 0\}.$$

The (relative) hull-kernel closure of E is the set

$$\text{hull ker } E = \{a \in X : f(a) = 0 \text{ whenever } f \in \ker E\}.$$

(1.2) Lemma. *Let $E \subset X$ and let $A \mid E$ be closed. Then* hull ker E *equals the A-convex hull of E, and $A \mid F$ is closed whenever $E \subset F \subset$* hull ker E.

PROOF. By the open mapping theorem, $A \mid E$ is closed if and only if there exists $M > 0$ such that each $f \in A \mid E$ has an extension $g \in A$ such that $\|g\|_X \leqslant M \|f\|_E$. Let $A \mid E$ be closed, let M be so chosen, and let $E \subset F \subset$ hull ker E. Let $f \in A$. Then $f \mid E$ has an extension $g \in A$ such that $\|g\|_X \leqslant M \|f\|_E$. Since $f = g$ on E, it follows that $f = g$ on F, so g extends $f \mid F$ and $\|g\|_X \leqslant M \|f\|_F$. Thus $A \mid F$ is closed.

In general, hull ker E contains the A-convex hull of E.

If $A \mid E$ is closed, if M is as above, and $F =$ hull ker E, then clearly $\|f\|_F \leqslant M \|f\|_E$ for all $f \in A$. Replacing f by f^n and taking roots and limits we conclude that $\|f\|_F \leqslant \|f\|_E$, so that F is contained in A-convex hull of E. This completes the proof.

(1.3) Corollary. *Let $E \subset X$, and let F be the A-convex hull of E. Then $A \mid E$ is closed if and only if $A \mid F$ is closed.*

(1.4). In view of this result, we focus on the family

$$\mathscr{R} = \{E \subset X : E \text{ is } A\text{-convex and } A \mid E \text{ is closed}\}.$$

We would like to give 'explicit' conditions for $E \in \mathscr{R}$. We would also like to find any structure in the family $\mathscr{R}$. We are particularly interested in the cases when $X = M(A)$, the maximal ideal space of A, and when $X = \text{Ш}(A)$, the Shilov boundary of A.

Glicksberg [5] showed that each p-set belongs to $\mathscr{R}$. He showed the converse in case A is logmodular and $X = \text{Ш}(A)$. Bernard [1] showed the converse in case A has unique representing measures on $X = \text{Ш}(A)$ and $M(A)$ is 'bien-partagé' (the representing measure for each point of each Gleason part P of A is supported on the weak-star closure of P in $M(A)$). These results apply to many examples, but they fail to cover such simple algebras as $A(U)$, where U is an annulus. In (2.1) we prove a result which covers $A(U)$ whenever U is a plane domain bounded by a finite number of closed curves, and which also covers some infinitely-connected U.

As regards structure, examples show that the family $\mathscr{R}$ is not in general closed under unions or intersections. We do however, have the following.

(1.5) Theorem. *Let $X = M(A)$, and suppose E and F are disjoint A-convex sets such that $A \mid E$ and $A \mid F$ are closed. Then $A \mid (E \cup F)$ is closed.*

Observe that E and F are hulls, by Lemma (1.2), so that $E \cup F$ is a hull. By the Shilov idempotent theorem, there is a function $h \in A$ such that $h = 0$ on E while $h = 1$ on F. If now $f \in A \mid F \cup E$, set $f_1 = f \mid E$ and $f_2 = f \mid F$, and let $F_1, F_2 \in A$ be extensions of f_1 and f_2, respectively, such that $\| F_j \| \leqslant c \|f_j\|$. Then $F = hF_2 + (1 - h)F_1$ extends f and $\| F \| \leqslant c(\| h \| + \| 1 - h \|) \|f\|$.

2. Main result

(2.1) Theorem. *Let A be a uniform algebra on $X \subset M(A)$. Suppose* (1) *A has no completely-singular annihilating measures on X,* (2) *for each $a \in M(A)$, each representing measure for a on X is supported on the hull-kernel closure in $M(A)$ of the Gleason part of a, and* (3) *for each $a \in M(A)$ and each representing measure ν for a on X, there exists a Jensen measure μ for a on X, with $\nu \ll \mu$. Let E be A-convex. Then $A|E$ is closed if and only if $\mu(E) = 0$ or 1 for each Jensen measure μ on X.*

Proof. Glicksberg [5; 2, p. 58] showed that a closed set E is a p-set if and only if $\mu \mid E \in A^{\perp}$ whenever $\mu \in A^{\perp}$. Bernard [1, p. 377] deduced that if E is a p-set, then E is *ergodic* for all representing measures for A on X, in the sense that if μ is such a measure, then $\mu(E)$ is 0 or 1. If A has no completely-singular annihilating measures, then by the general F. and M. Riesz theorem [2, p.45, (7.11)], if E is ergodic for all representing measures, then E is a p-set. Thus in the present situation, E is a p-set if and only if E is ergodic for all representing measures, or equivalently for all Jensen measures. The 'if' part of the result is now clear.

Suppose E is a hull. We wish to show that E is a p-set. It suffices to take $E = f^{-1}(0)$ for some $f \in A$. Let τ be a representing measure for a point $a \in M(A)$. We wish to show that $\tau(E) = 0$ or 1. Suppose $\tau(E) > 0$. Let $P \subset M(A)$ be the Gleason part of a, and let b be any point of P. Then b has a representing measure ν on X such that $\tau \ll \nu$ [2, p. 143, (1.2)], hence $\nu(E) > 0$. By hypothesis (3), there is a Jensen measure μ for b with $\nu \ll \mu$, hence $\mu(E) > 0$, so that

$$\log |\hat{f}(b)| \leqslant \int \log |f| \, d\mu = -\infty,$$

where $\hat{f}$ denotes the Gelfand transform of f. Thus $\hat{f}(b) = 0$. Thus $\hat{f} = 0$ on P, hence $\hat{f} = 0$ on the hull-kernel closure of P in $M(A)$. By hypothesis (2), $\hat{f} = f = 0$ on the support of τ, hence $\tau(E) = 1$. Thus each hull is a p-set, and the result follows from Lemma (1.2).

(2.2) This result does not imply Glicksberg's result for logmodular algebras. The hypothesis on the support of representing measures fails for any algebra with a one-point part off the Shilov boundary. It would be interesting to know if each $E \in \mathscr{H}$ is a p-set whenever A is hypodirichlet on $X = ш(A)$ (cf. (2.4) below).

(2.3) Let U be an open subset of the Riemann sphere, and let $A(U)$ denote the algebra of all functions continuous on clos U and analytic on U. Suppose $A(U)$ has nonconstant functions in it. The maximal ideal space of $A(U)$ is clos U (Arens' theorem). The Shilov boundary of $A(U)$ is the *essential boundary* of U, i.e. the set of points $a \in bdy\ U$ such that a is an essential singularity for some $f \in A(U)$. The algebra $A(U)$ has no completely-singular annihilating measures, and representing measures for a point of clos U are always supported on the closure of the part [4]. Thus we obtain the following.

Corollary. *Let $A = A(U)$ on $X = ш(A)$, and suppose that each representing measure for each point $a \in$ clos U is absolutely-continuous with respect to some Jensen measure for a. Let $E \subset X$ be closed. Then $A \mid E$ is closed if and only if E is ergodic for all Jensen measures on X.*

Note that in this case *all* closed subsets of X are A-convex.

(2.4). If A is hypodirichlet on X, then $X = ш(A)$ and all representing measures on X for a point $a \in M(A)$ are dominated by the (unique) Jensen measure for a [2, p. 116]. Thus the corollary applies to all hypodirichlet $A(U)$. For such $A(U)$, the Jensen measure for $a \in U$ is in fact the harmonic measure for a on the boundary of the component of a. Thus $A(U) \mid E$ is closed if and only if E is ergodic for all these harmonic measures.

Gamelin and Garnett [4] gave fairly explicit necessary and sufficient conditions on U for $A(U)$ to be hypodirichlet.

As an example, if U is bounded by n smooth curves, then $A(U)$ is hypodirichlet, and each harmonic measure is mutually absolutely-continuous with respect to arc length on its support.

The fact that each $E \in \mathscr{H}$ is a p-set for hypodirichlet $A(U)$ does not follow from Glicksberg's result on logmodular algebras. Indeed $A(U)$ is logmodular if and only if it is dirichlet.

Corollary (2.3) applies to some non-hypodirichlet $A(U)$, for instance, certain 'champagne bubble' algebras [2, p. 227].

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