INSTABILITY PAIRS

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ABSTRACT

Let μ be a capacity on $\mathbb{R}^d$, i.e. a translation-invariant non-negative non-decreasing Borelian set function. We shorten $\mu B(a, r)$ to $\mu(r)$. Let λ be a content on $\mathbb{R}^d$, i.e. a countably quasi-subadditive capacity, and asssume $\lambda(r)/\mu(r) \to 0$ as $r \downarrow 0$.

Conditions are given on (λ, μ) which are sufficient to ensure that each of the following hold, for each Borel set E and λ almost all $a \in \mathbb{R}^d$. (We abbreviate $E \cap B(a, r) = E(a, r)$.)

$$(A) \quad \limsup_{r\downarrow 0} \frac{\mu E(a, r)}{\mu(r)} > 0 \quad \text{or} \quad \limsup_{r\downarrow 0} \frac{\mu E(a, r)}{\lambda(r)} < +\infty,$$

$$(B) \quad \liminf_{r\downarrow 0} \frac{\mu E(a,r)}{\mu(r)} > 0 \quad \text{or} \quad \lim_{r\downarrow 0} \frac{\mu E(a,r)}{\mu(r)} = 0,$$

$$(C) \quad \limsup_{r\downarrow 0} \frac{\mu E(a, r)}{\lambda(r)} = +\infty \quad \text{or} \quad \lim \frac{\mu E(a, r)}{\lambda(r)} = 0.$$

The results are applied to specific examples.

§1. An instability theorem is a bit like a density theorem. The ordinary density theorem for Lebesgue measure $\mathscr{L}^d$ states that, given a Borel set $E \subset \mathbb{R}^d$, we have

$$\lim_{r\downarrow 0} \frac{\mathscr{L}^d(E \cap B(a, r))}{\mathscr{L}^d B(a, r)} = 1$$

for $\mathscr{L}^d$ almost all $a \in E$. The density theorem for α-dimensional Hausdorff content (where $0 < \alpha < d$) [2, (2.10.19)(2)] has

$$\limsup_{r\downarrow 0} \frac{M^\alpha(E \cap B(a, r))}{r^\alpha} \geqslant \kappa_1 \tag{1}$$

for M^α almost all $a \in E$, where $\kappa_1 = \kappa_1(\alpha, d) > 0$. The instability theorem (proved below) for the pair $(\mathscr{L}^d, M^\alpha)$, states that at $\mathscr{L}^d$ (*not* M^α) almost all points $a \in \mathbb{R}^d$ at which (1) fails, we have

$$\limsup_{r\downarrow 0} \frac{M^\alpha(E \cap B(a, r))}{r^2} = 0. \tag{2}$$

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Here are some classical examples of instability theorems.

Vitushkin's theorem [13, p. 190] (let γ denote the (outer) analytic capacity of Ahlfors). *Given a Borel set $E \subset \mathbb{C}$, we have*

$$\lim_{r\downarrow 0} \frac{\gamma(E \cap B(a, r))}{r} = 1 \text{ or } \lim_{r\downarrow 0} \frac{\gamma(E \cap B(a, r))}{r^2} = 0$$

for $\mathscr{L}^2$ almost all $a \in \mathbb{C}$.

The Lysenko-Pisarevskii theorem [9] (a weaker version was first found by Gonchar; let C denote harmonic capacity on $\mathbb{R}^3$). *Given a Borel set $E \subset \mathbb{R}^3$, we have*

$$\lim_{r\downarrow 0} \frac{C(E \cap B(a, r))}{r} = 1 \text{ or } \lim_{r\downarrow 0} \frac{C(E \cap B(a, r))}{r^3} = 0$$

for $\mathscr{L}^3$ almost all $a \in \mathbb{R}^3$.

Gonchar's theorem [7, theorem 2, p. 161] (let Cap denote logarithmic capacity on $\mathbb{C}$). *Given an open dense set $E \subset \mathbb{C}$, we have*

$$\liminf_{r\downarrow 0} \frac{\operatorname{Cap}(E \cap B(a, r))}{\log \frac{1}{r}} > 0 \text{ or } \lim_{r\downarrow 0} \frac{\operatorname{Cap}(E \cap B(a, r))}{r^2} = 0$$

for $\mathscr{L}^2$ almost all $a \in \mathbb{C}$.

Melnikov's theorem [10, p. 123]. *Let E be an open set in $\mathbb{C}$, and $1 < \beta < 2$. Then*

$$\limsup_{r\downarrow 0} \frac{\gamma(E \cap B(a, r))}{r} > 0 \text{ or } \limsup_{r\downarrow 0} \frac{\gamma(E \cap B(a, r))}{r^\beta} < +\infty$$

for M^β almost all $a \in \mathbb{C}$.

Hedberg's theorem [8, p. 309, theorem 9] (Hedberg has a capacity $C_{K,q}$ on $\mathbb{R}^d$, corresponding to a convolution kernel K and an index $q \in [1, \infty]$). *Let $E \subset \mathbb{R}^d$ be a dense Borel set. Then the following two conditions are equivalent:*

(*a*) $C_{K,q}(E \cap B) = C_{K,q}(B)$ *for all balls B;*

(*b*) $\limsup_{r\downarrow 0} \frac{C_{K,q}(E \cap B(a, r))}{r^d} > 0$

for $\mathscr{L}^d$ almost all $a \in \mathbb{R}^d$.

Polking's theorem [12, theorem (2.6)] (let $b_{m,q}$ denote the L_q Bessel potential of order m on $\mathbb{R}^d$). *Let $E \subset \mathbb{R}^d$ be a dense open set. Then*

$$\inf_{a,r} \frac{b_{m,q}(E \cap B(a, r))}{b_{m,q}B(a, r)} > 0$$

if and only if

$$\limsup_{r \downarrow 0} \frac{b_{m,q}(E \cap B(a, r))}{r^d} > 0$$

for $\mathscr{L}^d$ almost all $a \in \mathbb{R}^d$.

You get the idea.

§ **2.** For the purposes of this paper, a *capacity* on $\mathbb{R}^d$ is a translation-invariant non-negative non-decreasing Borelian set function on $\mathbb{R}^d$, i.e. μ is a capacity if

(1) $\mu: 2^{\mathbb{R}^d} \to [0, \infty]$,

(2) $E \subset F \Rightarrow \mu E \leqslant \mu F$,

(3) $\mu(a+E) = \mu E$, $\forall\, a \in \mathbb{R}^d$, $\forall\, E \subset \mathbb{R}^d$,

(4) $(a, r) \to \mu(E \cap B(a, r))$ is a Borel function, whenever E is Borel.

A capacity λ is a *content* if it is countably quasi-subadditive, i.e. there exists $\kappa_2(\lambda) > 0$ such that

$$\lambda \bigcup_{n=1}^{\infty} E_n \leqslant \kappa_2 \sum_{n=1}^{\infty} \lambda(E_n)$$

whenever E_n is a Borel set for $n = 1, 2, 3, \ldots$. If μ is a capacity, we abbreviate $\mu B(a, r)$ to $\mu(r)$, abusing the notation. We say that (λ, μ) is a *pair* on $\mathbb{R}^d$ if λ is a content on $\mathbb{R}^d$, μ is a capacity on $\mathbb{R}^d$, and $\lambda(r)/\mu(r) \to 0$ as $r \downarrow 0$. The capacities of practical interest fall, up to bounded equivalence, into three (overlapping) groups.

(1) *Analytic capacities* in the sense of Dolženko. Given a norm $\|\cdot\|_F$ on $\mathbb{C}^{S^2}$, the capacity $\gamma_F(E)$ of a compact set E is $\sup_f |a_1(f)|$ where f runs through all functions analytic off E, with $\|f\|_F \leqslant 1$, and $f(z) = a_0 + a_1/z + \ldots$ near ∞. For open U, $\lambda_F(U) = \sup\{\gamma_F(E): E \subset U, E \text{ compact}\}$. For arbitrary E, $\gamma_F(E) = \inf\{\gamma_F(U): E \subset U, U \text{ open}\}$. The analytic capacity γ corresponds to the sup norm, and the analytic p-capacity γ_p corresponds to the $L_p(\mathscr{L}^2)$ norm.

(2) *Kernel capacities.* Given a continuous function $K: \mathbb{R}^{2d} \to (0, \infty)$ and a number $p \in [1, \infty]$, the capacity $C_{K,p}(E)$ of a Borel set E is $\sup_\mu \mu E$ where μ runs over all positive Borel measures on E such that the potential

$$U^\mu(x) = \int K(x,y)\, d\mu(y)$$

belongs to the unit ball of $L_p(\mathscr{L}^d)$. For arbitrary $E \subset \mathbb{R}^d$, $C_{K,p}(E) = \inf\{C_{K,p}(B): E \subset B,$ B is Borel$\}$. The Frostman capacities C^α [4, 3], including Newtonian capacity C, are examples, as are the more general capacities $C_{K,q}$ and $b_{s,q}$ used by Hedberg and Polking. The logarithmic capacity Cap fits a minor variation of the definition — namely replace 'positive measures' by 'measures of total mass zero' [4].

(3) *Hausdorff contents.* These are contents such that

$$\mu(E) = \inf_{\mathscr{S}} \sum_{B \in \mathscr{S}} \mu B$$

where $\mathscr{S}$ runs over all countable coverings of E by balls. The contents M^α and $\mathscr{L}^d$ are examples.

The capacities $\mathscr{L}^d$, γ_p, Cap, M^β, C^α are all contents, with the possible exception of $\gamma_\infty = \gamma$. It is an open problem whether γ is quasi-subadditive. However, $\mu = \gamma$ does have the following weak quasi-subadditivity property, which we call property (W):

there exists a constant $\kappa_2(\mu) > 0$ *such that*

$$\mu(E) \leqslant \kappa_2 \sum_{B \in \mathscr{S}} \mu(E \cap B)$$

wherever $\mathscr{S}$ *is a countable covering of* E *by (closed) dyadic cubes.*

This property, for $\mu = \gamma$, is a consequence of Melnikov's estimate [5, p. 230].

Consider the following properties that a pair (γ, μ) might have, for each Borel set E:

$$(A) \quad \limsup_{r \downarrow 0} \frac{\mu(E \cap B(a, r))}{\mu(r)} > 0 \quad \text{or} \quad \limsup_{r \downarrow 0} \frac{\mu(E \cap B(a, r))}{\lambda(r)} < +\infty$$

for λ almost all $a \in \mathbb{R}^d$;

$$(B) \quad \liminf_{r \downarrow 0} \frac{\mu(E \cap B(a, r))}{\mu(r)} > 0 \quad \text{or} \quad \lim_{r \downarrow 0} \frac{\mu(E \cap B(a, r))}{\mu(r)} = 0$$

for λ almost all $a \in \mathbb{R}^d$;

$$(C) \quad \limsup_{r \downarrow 0} \frac{\mu(E \cap B(a, r))}{\lambda(r)} = +\infty \quad \text{or} \quad \lim_{r \downarrow 0} \frac{\mu(E \cap B(a, r))}{\lambda(r)} = 0$$

for λ almost all $a \in \mathbb{R}^d$;

$$(D) \quad \lim_{r \downarrow 0} \frac{\mu(E \cap B(a, r))}{\mu(r)} = 1 \quad \text{or} \quad \liminf_{r \downarrow 0} \frac{\mu(E \cap B(a, r))}{\mu(r)} = 0$$

for λ almost all $a \in \mathbb{R}^d$.

We say that (λ, μ) is an *instability pair* if it has property (A). If, in addition, it has one or more of the other properties, we call it a B-instability pair, a C-instability pair, a BC-instability pair, etc., as appropriate. For instance, if (λ, μ) is a BC-instability pair, then we have that

$$\liminf_{r \downarrow 0} \frac{\mu(E \cap B(a, r))}{\mu(r)} > 0 \quad \text{or} \quad \lim_{r \downarrow 0} \frac{\mu(E \cap B(a, r))}{\lambda(r)} = 0$$

for λ almost all a, whereas if (λ, μ) is a *BCD*-instability pair, then we have the dichotomy

$$\lim_{r\downarrow 0} \frac{\mu(E \cap B(a, r))}{\mu(r)} = 1 \quad \text{or} \quad \lim_{r\downarrow 0} \frac{\mu(E \cap B(a, r))}{\lambda(r)} = 0$$

for λ almost all a.

Summary theorem.

(α) *(M^β, γ) for $1 < \beta < 2$, (M^β, M^α) for $0 < \alpha < \beta < d$, (M^β, C^α) for $0 < \alpha < \beta < d$, (M^β, γ_p) for $2 < p < \infty$ and $\frac{p-2}{p} < \beta < 2$, are instability pairs.*

(β) *$(\mathscr{L}^2, \gamma_p)$, $(\mathscr{L}^d, C_{K,q})$, $(\mathscr{L}^d, b_{m,q})$, $(\mathscr{L}^d, M^\alpha)$, and $(\mathscr{L}^d, C^\alpha)$ are BC-instability pairs.*

(γ) *$(\mathscr{L}^2, \gamma)$ and $(\mathscr{L}^3, C)$ are BCD-instability pairs.*

In the remainder of the paper, we discuss properties (A)-(D) in turn, and explain how to go about proving (α), (β), (γ). First, in § 3, we present and prove a version of Melnikov's covering lemma. In § 4 we discuss instability and B-instability. In §§5-6 we deal with properties (C) and (D).

To date, most of the applications of instability theorems have been in connection with approximation theorems. Essentially, an instability theorem allows us to relax a hypothesis of the form

$$\lim_{r\downarrow 0} (\sup/\inf) \frac{\mu(E \cap B(a, r))}{\mu(r)} > 0 \qquad \lambda\text{-a.e. on } F$$

to

$$\lim_{r\downarrow 0} (\sup/\inf) \frac{\mu(E \cap B(a, r))}{\lambda(r)} > 0 \qquad \lambda\text{-a.e. on } F.$$

Alternatively, it allows us to strengthen a conclusion of the latter form. There are examples in the papers already cited. As another example, take the *BC*-instability of the pair $(\mathscr{L}^d, M^\alpha)$. In a future paper, it will be shown that for $0 < \alpha < 1$ and $p > \frac{2}{1-\alpha}$, a function $f \in W^{1,p}_{loc}(\mathbb{C})$ (a Sobolev space) is a Lip α limit on the compact $\underline{X} \subset \mathbb{C}$ of rationals, if and only if $\frac{\partial f}{\partial \bar{z}}(a) = 0$ at $\mathscr{L}^2$ almost all points $a \in \underline{X}$ at which

$$\lim_{r\downarrow 0} \frac{M^{1+\alpha}(B(a, r) \sim \underline{X})}{r^{1+\alpha}} = 0.$$

Applying the instability, this condition is equivalent to the *a priori* weaker condition that $\frac{\partial f}{\partial \bar{z}}(a) = 0$ at $\mathscr{L}^2$ almost all points $a \in \underline{X}$ such that

$$\lim_{r\downarrow 0}\frac{M^{1+\alpha}(B(a,r)\sim \underline{X})}{r^2}=0.$$

§3. We say that a capacity μ on $\mathbb{R}^d$ is *approximately homogeneous* if

$$\sup_{r>0}\frac{\mu(rS)}{\mu(2rS)}<1,$$

for any (or, equivalently, each) cube S. The capacities $\mathscr{L}^d$, M^β, C^β, $\gamma_p(2<p<\infty)$, γ, C are approximately homogeneous; but Cap is not.

Following [11, p. 405], we say that a family $\mathscr{S}$ of balls (resp., dyadic cubes) is (μ,κ)-*invulnerable* if

$$\sum_{\substack{B\in\mathscr{S}\\ B\subset D}}\mu B\leqslant \kappa.\mu D$$

for each ball (resp., dyadic cube) D.

Melnikov's covering lemma (cf. [10, lemma 2; 6, lemma 4.5; 2, (2.8), for $\varphi=\mathscr{L}^d$; 9, lemma 1.1; 11, p. 405]). *Let μ be an approximately homogeneous capacity on $\mathbb{R}^d$, having property (W). Then there exist constants $\kappa_3>0$ and $\kappa_4>0$, depending only on μ and d such that each bounded family $\mathscr{S}$ of balls of positive radii contains a (μ,κ_3)-invulnerable subfamily $\mathscr{S}_1$ such that*

$$\sum_{B\in\mathscr{S}_1}\mu B\geqslant \kappa_4\mu\left(\bigcup\mathscr{S}\right).$$

PROOF. It suffices to prove the dyadic analogue, since μ is non-decreasing and has property (W).

Let $\mathscr{S}$ be a bounded family of dyadic cubes. Let

$$\alpha=\sup_{r>0}\frac{\mu(rS)}{\mu(2rS)}$$

and let $\kappa_5=(1-\alpha)^{-2}$, $\kappa_6=\kappa_2(\mu)^{-1}$. We shall construct a subfamily $\mathscr{S}_1\subset\mathscr{S}$ which is (μ,κ_5)-invulnerable and satisfies

$$\sum_{B\in\mathscr{S}_1}\mu B\geqslant \kappa_6\mu\left(\bigcup\mathscr{S}\right).$$

Step 0. Let $\kappa_7=(1-\alpha)^{-1}$. If $\mathscr{S}$ is (μ,κ_7)-invulnerable, take $\mathscr{S}_1=\mathscr{S}$. Otherwise, choose a dyadic cube D'_1 of maximal side such that

$$\sum_{\substack{B\in\mathscr{S}\\ B\subset D'_1}}\mu B>\kappa_7\mu D'_1.$$

This is possible because the number of candidates of a given side is necessarily finite, and is zero for large enough side. If $\mathscr{S} \sim 2D^{D'_1}$ is (μ, κ_7)-invulnerable, then stop (this step). Otherwise, choose D'_2 of maximal side such that

$$\sum_{\substack{B \in \mathscr{S} \sim 2^{D'_1} \\ B \subset D'_2}} \mu B > \kappa_7 \mu(D'_2).$$

Necessarily, $\mu D'_2 \leqslant \mu D'_1$, and the interiors of D'_1 and D'_2 are disjoint (for dyadic cubes this last is equivalent to saying that neither contains the other). Continue. If the process does not stop, then $\mu D'_n \downarrow 0$, since the number of cubes of each side which might become D'_ns is finite. Thus, if a cube $B \in \mathscr{S}$ is not removed in some $2^{D'n}$ before side(D'_n) falls below side(B), then it is not removed at all. Let D_n be the least dyadic cube containing $\mathscr{S} \cap 2^{D'}_n$. The cubes D_n are the *first generation composites*, and the cubes of $\mathscr{S} \sim \bigcup_n 2^{D_n}$ are the *first generation integrals.* Together, these composites and integrals make up the whole first generation, which is a (μ, κ_7)-invulnerable family. Set the *initial cluster factor* $f(D_n)$ of each composite equal to 1, and the *cluster factor* $k(B)$ of each integral equal to 1.

Step n. Suppose that D is an nth generation composite cube with initial cluster factor $f(D) \in [1-\alpha, 1]$. Choose a subfamily $\mathscr{T} \subset \mathscr{S} \cap 2^D$ such that

$$f(D)\mu D \leqslant \sum_{B \in \mathscr{T}} \mu B < \left\{ f(D) + \frac{\mu(\frac{1}{2}D)}{\mu(D)} \right\} \mu D.$$

This is possible since

$$\sum_{\substack{B \in \mathscr{S} \\ B \subset D}} \mu B > \mu D$$

and the cubes belonging to $\mathscr{S} \cap 2^D$ have at most half the side of D. The family $\mathscr{T} = \mathscr{T}(\mathrm{D})$ is the $(n+1)$*st generation cluster* associated with D.

Repeat step 0, with $\mathscr{S}$ replaced by $\mathscr{T}$. The composites D_n are called $(n+1)$st *generation composites*, and the integrals, $(n+1)$st *generation integrals.* Note that

$$\sum_{B \in \mathscr{T}} \mu B < (1+\alpha)\mu D < \kappa_7 \mu D,$$

so D itself is not composite this time. The *final cluster factor* $h(D)$ of D is $f(D).\mu D / \sum_{B \in \mathscr{T}} \mu B$. This is also the initial cluster factor of the new D_n, and the cluster factor of the new integrals. Note that $h(D) \in [1-\alpha, 1]$.

The process may stop after a finite number of steps, or continue indefinitely. Since an $(n+1)$st generation composite is always smaller than the nth generation composite that contains it, each element of $\mathscr{S}$ either becomes an integral cube or is dropped in the formation of some cluster. The integral cubes of all generations form the family $\mathscr{S}_1$.

If D is a composite (of any generation), then a calculation shows that

$$f(D)\mu D = \sum_{\substack{B \in \mathscr{S}_1 \\ B \subset D}} k(B)\mu B.$$

Thus

$$\begin{aligned} \mu\left(\bigcup \mathscr{S}\right) &\leqslant \kappa_2 \sum_{C \text{ first generation}} \mu C \\ &= \kappa_2 \sum_{B \in \mathscr{S}_1} k(B)\mu B \\ &\leqslant \kappa_2 \sum_{B \in \mathscr{S}_1} \mu B. \end{aligned}$$

Let D be any dyadic cube. If D contains any composites, take m minimal such that D contains an mth generation composite. Each composite C contained in D is contained in some mth generation composite, so we compute

$$\begin{aligned} \mu D &\geqslant \kappa_7^{-1} \sum_{\substack{B \subset D \\ B\ m^{\text{th}}\ \text{gen.}}} \mu B \\ &= \kappa_7^{-1} \left\{ \sum_{\substack{B \subset D \\ B\ m^{\text{th}}\ \text{gen.} \\ B\ \text{integral}}} \mu B + \sum_{\substack{B \subset D \\ B\ m^{\text{th}}\ \text{gen.} \\ B\ \text{composite}}} f(B)^{-1}. \sum_{\substack{B' \in \mathscr{S}_1 \\ B' \subset B}} k(B')\mu B' \right\} \\ &\geqslant \kappa_7^{-2} \sum_{\substack{B \subset D \\ B \in \mathscr{S}_1}} \mu B', \end{aligned}$$

as required.

§4. We say that a content λ on $\mathbb{R}^d$ has a *density theorem* (resp., *strong density theorem*) if, given a Borel set $E \subset \mathbb{R}^d$,

$$\limsup_{r \downarrow 0} \text{ (resp., } \liminf_{r \downarrow 0}) \frac{\lambda(E \cap B(a, r))}{\lambda(r)} > 0,$$

for λ almost all $a \in E$. So $\mathscr{L}^d$ has a strong density theorem, and M^α has a density theorem. However, M^α does not have a strong density theorem [2, (3.3.21)].

We say that the pair (λ, μ) has property (E) if there exists κ_8: $(1, \infty) \to (0, \infty)$ such that given a (λ, σ_1)-invulnerable family $\mathscr{S}$ of balls, with $\sigma_1 > 1$, and given a Borel set $E \subset \mathbb{R}^d$ and a number $\delta \geqslant \operatorname{diam} E$ such that

$$\mu(E \cap B) > \frac{\mu(\delta)}{\lambda(\delta)}. \lambda B, \quad \forall\, B \in \mathscr{S},$$

it then follows that

$$\mu E \geqslant \kappa_8(\sigma_1) . \frac{\mu(\delta)}{\lambda(\delta)} \sum_{B \in \mathscr{S}} \lambda B.$$

This property is a mild kind of superadditivity.

Instability lemma. *Let (λ, μ) be a pair on $\mathbb{R}^d$ such that λ is approximately homogeneous and has a density theorem (resp., strong density theorem). Suppose (λ, μ) has property (E). Then (λ, μ) is an instability pair (resp. a B-instability pair).*

PROOF. Fix a Borel set $E \in \mathbb{R}^d$. Let E_0 denote the (Borel) set of those $a \in \mathbb{R}^d$ at which

$$\limsup_{r \downarrow 0} \frac{\mu(E \cap B(a, r))}{\lambda(r)} = +\infty.$$

At λ almost all points a of E_0, we have

$$\limsup_{r \downarrow 0} \frac{\lambda(E_0 \cap B(a, r))}{\lambda(r)} > 0$$

$$\text{(resp., } \liminf_{r \downarrow 0} \frac{\lambda(E_0 \cap B(a, r))}{\lambda(r)} > 0).$$

Let a be such a point. We shall prove that

$$\limsup_{r \downarrow 0} \text{ (resp., } \liminf_{r \downarrow 0}) \frac{\mu(E \cap B(a, r))}{\mu(r)} > 0.$$

This will suffice. Fix $\tau > 0$ and $\delta > 0$ with

$$\lambda(B(a, \delta) \cap E_0) > \tau\lambda(\delta).$$

We shall prove that

$$\mu(B(a, \delta) \cap E) \geqslant \kappa_9 \tau \mu(\delta),$$

where κ_9 depends only on μ and λ, and not on δ. This will suffice.

The set $E_1 = E_0 \cap B(a, \delta)$ may be covered by a bounded family $\mathscr{S}$ of balls such that

$$B \in \mathscr{S} \Rightarrow \mu(E \cap B) > \frac{\lambda B . \mu(\delta)}{\lambda(\delta)}.$$

By Melnikov's covering lemma, there is a (λ, κ_3)-invulnerable subfamily $\mathscr{S}_1$ such that

$$\sum_{B \in \mathscr{S}_1} \lambda B \geqslant \kappa_4 \lambda(\bigcup \mathscr{S}) \geqslant \kappa_4 \tau \lambda(\delta).$$

Applying property (E), with $\sigma_1 = \kappa_3$, we conclude that

$$\mu(E \cap B(a, \delta)) \geqslant \kappa_8(\kappa_3)\kappa_4\tau\mu(\delta),$$

so we have the promised estimate, with $\kappa_9 = \kappa_8(\kappa_3)\kappa_4$, and the lemma is proved.

The instances of instability that have not hitherto been completely proved in print and that are asserted in the summary theorem are (M^β, γ), (M^β, M^α), (M^β, C^α), (M^β, γ_p), (M^β, Cap), $(\mathscr{L}^d, M^\alpha)$, and $(\mathscr{L}^d, C^\alpha)$. Since $\mathscr{L}^d = M^d$, $\gamma = \gamma_\infty$, and Cap is boundedly equivalent to γ_2^2, it boils down to the fact that we must establish property (E) for (M^β, M^α), (M^β, C^α), and (M^β, γ_p).

Lemma. (i) *Let* $0 < \alpha < \beta \leqslant d$. *Then* (M^β, M^α) *has property* (E).

(ii) *Let* $0 < \alpha < \beta \leqslant d$. *Then* (M^β, C^α) *has property* (E).

(iii) *Let* $2 \leqslant p \leqslant \infty$ *and* $\frac{p-2}{p} < \beta \leqslant d$. *Then* (M^β, γ_p) *has property* (E).

Proof. To begin with, we treat (i), (ii) and (iii) together, denoting M^α or C^α or γ_p by μ. Since μ has property (W), it suffices to prove the dyadic equivalent. So suppose $\mathscr{S}$ is an (M^β, σ_1)-invulnerable family of dyadic cubes, for some $\sigma_1 > 1$, and E is a Borel set of diameter $\leqslant \delta$, and $\mu(E \cap B) \geqslant \mu(\delta)\delta^{-\beta}M^\beta(B)$ for each $B \in \mathscr{S}$. We wish to show that

$$\mu E \geqslant \kappa_8(\sigma_1).\mu(\delta).\delta^{-\beta} \sum_{B \in \mathscr{S}} M^\beta(B).$$

We may take it that no cube in $\mathscr{S}$ contains another, for the invulnerability of $\mathscr{S}_1$ implies that

$$\sum_{\substack{B \in \mathscr{S} \\ B \text{ maximal}}} M^\beta(B) \geqslant \sigma_1^{-1} \sum_{B \in \mathscr{S}} M^\beta(B).$$

We may assume that E is open, since, in general,

$$\mu E = \inf \{\mu U : E \subset U,\ U \text{ open}\}.$$

We may assume that $\mathscr{S}$ is finite, since each subset of an (M^β, σ_1)-invulnerable family is (M^β, σ_1)-invulnerable. Let $E_B = E \cap B$.

Case (i), in which $\mu = M^\alpha$. For each $B \in \mathscr{S}$, there is a measure υ_B of *growth* α (i.e. $\upsilon_B D \leqslant M^\alpha D$, $\forall$ balls D), with $\upsilon_B(\mathbb{R}^d \sim E_B) = 0$, and $\|\upsilon_B\| \geqslant \kappa_{11}(d)M^\alpha(E_B)$. Let

$$\upsilon = \sum_{B \in \mathscr{S}} \frac{M^\beta(B)}{M^\alpha(E_B)} \upsilon_B.$$

Then $\|\upsilon\| \geqslant \kappa_{11} \sum M^\beta(B)$.

Fix a dyadic cube D. If $B \in \mathcal{S}$, at least one of three things occurs: (1) $B \cap D = \varnothing$, (2) B meets a face of D, (3) $B \subset 2D$. Possibility (2) occurs for at most $\kappa_{12}(d)$ cubes B, since no cube in $\mathcal{S}$ contains another. Thus

$$\begin{aligned} \upsilon D &= \sum_{(2)} \upsilon_B D + \sum_{(3)} \upsilon_B D \\ &\leqslant \kappa_{12}.\delta^{\beta-\alpha}.M^\alpha(D) + \sum_{(3)} M^\beta B \\ &\leqslant \kappa_{12}.\delta^{\beta-\alpha}.M^\alpha(D) + \sigma_1 M^\beta(2D), \end{aligned}$$

so for diam $D \leqslant 2\delta$ we have

$$\upsilon D \leqslant \kappa_{13}(\alpha, \beta, d, \sigma_1)\delta^{\beta-\alpha}.\ M^\alpha(D),$$

so that $\kappa_{13}^{-1}.\delta^{\alpha-\beta}.\upsilon$ has growth α, whence

$$M^\alpha(E) \geqslant \kappa_{13}^{-1}.\delta^{\alpha-\beta}.\kappa_{11}.\sum M^\beta(B),$$

as required.

Case (ii), in which $\mu = C^\alpha$. For each $B \in \mathcal{S}$, choose a positive Borel measure υ_B supported on a compact subset of E_B, with $\int |x-y|^{-\alpha} d\upsilon_B(y) \leqslant 1$ and $\|\upsilon_B\| \geqslant \frac{1}{2}C^\alpha(E_B)$. This is possible since C^α is a Choquet capacity [1]. (It is possible to avoid the use of this fact, using the method of the next case.) Let

$$\upsilon = \sum_{B \in \mathcal{S}} \frac{M^\beta(B)}{C^\alpha(E_B)}.\ \upsilon_B.$$

We have to show that for all $x \in \mathbb{R}^d$

$$P_\upsilon(x) = \int |x-y|^{-\alpha} d\upsilon(y) \leqslant \kappa_{14}\delta^{\beta-\alpha},$$

where $\kappa_{14} = \kappa_{14}(\alpha, \beta, d, \sigma_1)$.

Fix $x \in \mathbb{R}^d$. If dist $(x, E) > \delta$, we have $P_\upsilon(x) \leqslant \delta^{-\alpha}\|\upsilon\| \leqslant \delta^{-\alpha}\sum M^\beta(B) \leqslant \delta^{-\alpha}.\ \sigma_1.(2\delta)^\beta$, so that is all right. So suppose dist $(x, E) \leqslant \delta$. Abbreviate $\{M^\beta(B)/C^\alpha(E_B)\}\upsilon_B$ to τ_B. We have

$$P_{\tau_B}(x) \leqslant \min\left\{\delta^{\beta-\alpha}, \frac{M^\beta(B)}{\text{dist}\,(x, B)^\alpha}\right\},$$

so, using the invulnerability,

$$\begin{aligned} P_\upsilon(x) &\leqslant \kappa_{15}\left\{\delta^{\beta-\alpha} + \int_0^{3\delta} \frac{r^{\beta-1}dr}{r^\alpha}\right\} \\ &= \kappa_{16}\delta^{\beta-\alpha} \end{aligned}$$

where κ_{15} and κ_{16} depend only on α, β, d, and σ_1. That does it.

Case (*iii*), in which $\mu = \gamma_p$. To begin with, expand each cube B slightly, and replace σ_1 by $\frac{\sigma_1+1}{2}$, so that the expanded cubes B' still form an (M^β, σ_1)-invulnerable family. We can now find a compact set $K_B \subset E \cap \operatorname{int} B'$, with $\gamma_p(K_B) \geqslant \frac{1}{2}\gamma_p(E_B)$. For each B, take a function f_B, analytic off K_B, with $\|f_B\|_{L_p} \leqslant 1$, and $a_1(f_B) \geqslant \frac{1}{2}\gamma_p(E_B)$. Form

$$f = \sum_{B\in\mathcal{S}} \frac{M^\beta(B)}{\gamma_p(E_B)}.f_B.$$

Then $a_1(f) \geqslant \frac{1}{2}\sum M^\beta(B)$, so all we have to do is show that $\|f\|_{L_p} \leqslant \kappa_{17}(p, \beta, \sigma_1)\delta^\beta/\gamma_p(\delta)$.

For $p = \infty$, this estimate is practically the same as the case $\alpha = 1$ of case (ii) above, because of the inequality

$$|f_B(z)| \leqslant \frac{\gamma(K_B)}{\operatorname{dist}(z, K_B)}$$

[5, p. 201]. For $2 < p < \infty$, it is just a little more complicated. Use the estimate

$$|f_B(z)| \leqslant \kappa_{18}(p)\frac{\gamma_p(K_B)}{\operatorname{dist}(z, K_B)}$$

[7, p. 163], and bear in mind that $\gamma_p(\delta)$ is comparable to $\delta^{(p-2)/p}$.

The case of γ_2, or equivalently Cap ($\sim \gamma_2^2$), needs a special argument, as usual. The handiest thing is to work with the kernel capacity $\rho = C_{K,\infty}$, where $K(r) = 1+\log_+ r^{-1}$, which is comparable to Cap for sets of small diameter. Proceeding as in case (ii), pick υ_B on E_B with $P_\upsilon(x) = \int K(|x-y|)d\upsilon(y) \leqslant 1$ and $\|\upsilon_B\| \geqslant \frac{1}{2}\rho(E_B)$, form

$$\upsilon = \sum_{B\in\mathcal{S}} \frac{M^\beta(B)}{\rho(E_B)}.\upsilon_B,$$

and show that $\delta^{-\beta}\rho(\delta)P_\upsilon(x)$ is bounded, independently of δ, where $\rho(\delta)$ is essentially $(\log \delta^{-1})^{-1}$.

§5. Frostman [4, p. 57] called

$$\limsup_{r\downarrow 0} \frac{\mu E \cap B(a, r)}{\mu(r)}$$

the *upper μ capacitary density* of E at a. The capacities $\mu = \gamma_p(2 \leqslant p \leqslant \infty)$, C^α, Cap, $C_{K,q}$, $b_{q,s}$, M^α (as long as they form a pair with $\mathcal{L}^d$) all have the property (F): each $\mathcal{L}^d$ density point of E is a point of positive upper μ capacitary density of E. For instance, property (F) for M^α follows from the estimate $M^\alpha(E) \geqslant M^d(E)^{\alpha/d}$, which holds for all E, whereas for γ it follows from the estimate $\gamma(E) \geqslant \{\mathcal{L}^2(E)/\pi\}^{1/2}$.

It turns out that property (F) and the weak subadditivity (W) are enough to give the pair ($\mathcal{L}^d$, μ) property (C). This observation is essentially contained in [2, (2.9.17)].

Lemma. *If* ($\mathscr{L}^d$, μ) *is a pair and* μ *has properties* (*F*) *and* (*W*), *then* ($\mathscr{L}^d$, μ) *has property* (*C*).

PROOF. Fix a Borel set E. We have to show that

$$(1)\quad \limsup_{r\downarrow 0} \frac{\mu(E\cap B(a,r))}{r^d} = +\infty$$

or

$$(2)\quad \lim_{r\downarrow 0} \frac{\mu(E\cap B(a,r))}{r^d} = 0$$

for $\mathscr{L}^2$ almost all $a \in \mathbb{R}^d$.

If the upper $\mathscr{L}^d$ capacity density of E at a is positive, then by property (*F*) and the fact $\mu(r)/r^d \to +\infty$, we deduce (1). Thus it suffices to show that (2) holds at $\mathscr{L}^d$ almost all points a at which (1) fails and the $\mathscr{L}^d$ upper density of E is zero. Let G denote the set of such a. Then G is the union of the sets

$$G_n = \left\{ a \in \mathbb{R}^d : \mu E \cap B(a,r) \leqslant nr^d \text{ for } 0 < r < \frac{1}{n} \right\}$$

($n = 1,2,3,\ldots$), so it suffices to show that (2) holds at $\mathscr{L}^d$ almost all points of each G_n.

Fix n, $a \in G_n$, and $r \in \left(0, \frac{1}{2n}\right)$. There is a covering of $B(a, r) \sim G_n$ by a countable family $\mathscr{S}$ of balls B such that dist $(B, G_n) <$ diam B and $\sum$ (diam $B)^d \leqslant \kappa_{19}(d)\mathscr{L}^d(B(a,r) \sim G_n)$. This follows from Vitali's theorem.

Each ball $B \in \mathscr{S}$ is contained in a ball B' centred in G_n, with diam $B \leqslant$ diam $B' \leqslant$ 2 diam B, thus we obtain

$$\mu(E\cap B) \leqslant \mu(E \cap B') \leqslant n(\text{diam } B')^d \leqslant 2^d n\,(\text{diam } B)^d,$$

$$\mu(E \cap B(a,r)) \leqslant \kappa_2(\mu) \sum_{B\in\mathscr{S}} \mu(E\cap B) \leqslant \kappa_2.2^d.n.\kappa_{19}\mathscr{L}^d(B(a,r)\sim G_n).$$

Thus (2) holds at each point a of G_n at which $\mathbb{R}^d \sim G_n$ had $\mathscr{L}^d$ density zero, so it holds for $\mathscr{L}^d$ almost all points of G_n, as required.

§6. Finally, a word about property (*D*), the icing on the cake. We know of no 'intrinsic' proofs of property (*D*). Where it has been proved, it has been done by associating to the capacity μ an approximation problem. Typically, the problem concerns approximation in some norm by solutions of a partial differential equation. For example, for kernel capacities $C_{K,q}$, the elliptic equation has the kernel $K(|x|)$ for fundamental solution, and the norm is the $L_p(\mathscr{L}^d)$ norm. So far, the process has been one-way. Perhaps it is possible to start with an arbitrary capacity μ, and construct a partial differential operator L and norm $\|\cdot\|_F$ such that the capacity exactly characterises removable singularities and approximation for solutions of L, but this has not yet been done.

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