THE ORDER OF A SYMMETRIC CONCRETE SPACE

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ABSTRACT

We give a simple characterisation of those symmetric concrete spaces that inject locally into C^k on $\mathbb{R}^d$.

1. Introduction

The purpose of this paper is to establish a general theorem of Sobolev type. The theorem tells us when a function space on $\mathbb{R}^d$ imbeds into the space C^k. It applies to all reasonable function spaces, such as L^p spaces, Sobolev spaces, Besov spaces, Zygmund classes, Campanato spaces, and so on. Specifically, it applies to all spaces F in the class of *symmetric concrete Banach spaces*, defined below. The theorem identifies the one crucial property that is involved in deciding whether or not F imbeds in C^k. That property is the behaviour of the F-norm of monomials on small balls.

Let $\mathscr{D}$ denote the space $\mathscr{D}(\mathbb{R}^d, \mathbb{C})$ of complex-valued test functions on $\mathbb{R}^d$, and let $\mathscr{D}'$ denote its dual, the space of distributions on $\mathbb{R}^d$. With their usual topologies, these are topological vector spaces, and $\mathscr{D}'$ is a topological $\mathscr{D}$-module.

We say that a complete locally-convex topological vector space F is a symmetric concrete space ($F \in \mathrm{SCS}$) if it has the following four properties.

(1) $\mathscr{D} \subset F \subset \mathscr{D}'$, and the inclusions are continuous. Here, as is usual, we identify $\mathscr{D}$ with a subset of $\mathscr{D}'$.

(2) F is a topological sub-$\mathscr{D}$-module of $\mathscr{D}'$.

(3) $f \to \bar{f}$ is bicontinuous, from F onto F. Here, the complex conjugate $\bar{f}$ of the distribution f is defined by

$$\langle \phi, \bar{f} \rangle = \overline{\langle \bar{\phi}, f \rangle}, \ \forall \phi \in \mathscr{D}.$$

(4) For each $A \in \mathsf{Aff}$, the group of invertible affine transformation of $\mathbb{R}^d$, the map $f \to f \circ A$ is a continuous linear map of F into itself. Moreover, compact sets of affine transformations induce equicontinuous sets of endomorphisms of F. Here, the composition $f \circ A$ is defined by

$$\langle \phi, f \circ A \rangle = \langle \phi \circ A^{-1}, f \rangle \cdot \det(A)^{-1}, \ \forall \phi \in \mathscr{D}.$$

If an SCS is a Banach space, then we call it an SCBS.

Given $F \in$ SCS, we define the spaces

$$F_{\text{loc}} = \{ f \in \mathscr{D}' : \phi f \in F, \forall \phi \in \mathscr{D} \}$$
$$= \mathscr{E} \cdot F,$$

where $\mathscr{E}$ is the space of all infinitely-differentiable functions, and

$$F_{\text{cs}} = \mathscr{D} \cdot F = \{ f \in F : \operatorname{spt} f \text{ is compact} \}.$$

For X compact in $\mathbb{R}^d$, we consider the subspace

$$JF(X) = \operatorname{clos}_F \{ f \in F : X \cap \operatorname{spt} f = \emptyset \}$$

and we define $F(X)$ as the quotient space

$$F(X) = F/JF(X) ,$$

with the quotient topology. If F is Banach, Fréchet, or barrelled, then so is each $F(X)$. We may topologise F_{loc} and F_{cs}, in obvious ways, and they then become symmetric concrete spaces.

We say that two SCSs, F and G, are *locally equivalent* if $F_{\text{loc}} = G_{\text{loc}}$ as sets and as topological vector spaces. We use the notation $F \overset{\text{loc}}{\hookrightarrow} G$ to mean that $F_{\text{loc}} \subset G_{\text{loc}}$, and the inclusion is continuous. In other words, $F \overset{\text{loc}}{\hookrightarrow} G$ means that for each compact $X \subset \mathbb{R}^d$ there is a compact $Y \supset X$ such that the restriction $f \to f \mid X$ maps $F(Y)$ continuously into $G(X)$.

By C^k, $(k = 0, 1, 2, ...)$ we denote the Fréchet space of k times continuously differentiable functions $f: \mathbb{R}^d \to \mathbb{C}$. The spaces $\mathsf{C}^k(X)$ are Banach spaces, and C^k is locally equivalent to a Banach space, namely

$$\mathsf{BC}^k = \mathsf{C}^k \cap \{ f : \sup_{0 \leqslant j \leqslant k} |D^j f| \text{ is bounded on } \mathbb{R}^d \} .$$

2. Order

Let F be an SCBS. The function defined on $(0, \infty)$ by

$$\theta_k(r) : r \mapsto \| x_1^k \|_{F(B(0,r))} \qquad (k = 0, 1, 2, ...)$$

is increasing with r, and positive, so that $\lim_{r \downarrow 0} \theta_k(r) =_{\text{def}} \theta_k$ exists and is non-negative. If $\theta_k = 0$, then the $\mathscr{D}$-module property of F yields $\theta_{k+1} = 0$. Thus the set

$$\Sigma = \{ k \in \mathbb{Z}_+ : \theta_k \neq 0 \}$$

is $\emptyset$ or is an initial segment of $\mathbb{Z}_+$. We define

$$\text{order}(F) = \sup \Sigma .$$

Thus, order(F) may be $-\infty$, $+\infty$, or a non-negative integer. It is $-\infty$ when Σ is empty.

For instance, the reader may check that the order of C^k is k, the order of Lipα is 0 (for $0 < \alpha < 1$), the order of L^p is $-\infty$ for $p < +\infty$ and 0 for $p = +\infty$. For further examples, see section 3 below.

In the definition of order, it makes no difference if we replace 0 by some other point of $\mathbb{R}^d$. Nor does it matter if we replace the monomial x_1^k by any other homogeneous polynomial of degree k. These facts follow from the invariance of F under composition with elements of Aff.

If $F \overset{\text{loc}}{\hookrightarrow} G$, then order($F$) $\geqslant$ order(G). Thus the order depends only on the local equivalence class.

We can now state the theorem.

Theorem. *Let F be an SCBS and suppose order(F) $\geqslant k \geqslant 0$. Suppose also that $\mathscr{D}$ is dense in F. Then $F \overset{\text{loc}}{\hookrightarrow} \mathsf{C}^k$. Conversely, if $F \overset{\text{loc}}{\hookrightarrow} \mathsf{C}^k$, then order($F$) $\geqslant k$.*

The order of F depends only on $\text{clos}_F \mathscr{D}$, so the restriction that $\mathscr{D}$ be dense in F is essential in the first part. For instance, L^∞ has order 0.

PROOF. The converse is trivial, since $F \overset{\text{loc}}{\hookrightarrow} G$ implies order(F) $\geqslant$ order(G).

To prove the main assertion, suppose F has order at least k. We will show that $F \overset{\text{loc}}{\hookrightarrow} C^k$.

It suffices to consider the case when the order of F is exactly k. For otherwise we may pass to the (Banach!) space $F + \mathsf{BC}^k$.

For $a \in \mathbb{R}^d$, the relation $\overset{a}{\sim}$, defined by

$$f \overset{a}{\sim} g \Leftrightarrow \| f - g \|_{F(B(a,r))} \downarrow 0 \text{ as } r \downarrow 0$$

is an equivalence relation on F. For $\phi \in \mathscr{D}$, the equivalence class of ϕ contains exactly one polynomial of degree $\leqslant k$, namely $T_a^k \phi$, the kth order Taylor polynomial of ϕ about a.

The function

$$p \mapsto \inf_{r>0} \|p\|_{F(B(a,r))}$$

is a norm on $\mathbb{C}[x]_k$ (= the space of polynomials of degree at most k on $\mathbb{R}^d$), and hence is equivalent to any other norm on $\mathbb{C}[x]_k$.

Property (4) implies that bounded families of translations act equicontinuously on F, and hence the different norms corresponding to a bounded set of points a are uniformly equivalent.

For $f \in F$, if $\phi_n \in \mathscr{D}$ and $\|\phi_n - f\|_F \to 0$, then $\{T^k_a \phi_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}[x]_k$, and thus we may define a polynomial $T^k_a f$ by

$$T^k_a f = \lim_{n \uparrow \infty} T^k_a \phi_n \ .$$

This polynomial is independent of the choice of $\{\phi_n\}_1^{\infty}$ converging to f in F norm. Moreover, the $T^k_a \phi_n$ converge to $T^k_a f$ uniformly in a, provided a is restricted to any compact set. Thus, since the $T^k_a \phi_n$ are continuous in a, it follows that the $T^k_a f$ are continuous (i.e. continuously-varying polynomials in x) in a.

Define $g \in C^0$ by $g(a) = (T^k_a f)(a)$. When $\phi_n \to f$ in F, it follows that $\phi_n \to f$ in $\mathscr{D}'$. Since

$$\phi_n(a) = T^0_a \phi_n(a) \to T^0_a f(a) = g(a)$$

uniformly on compacta, it follows that $\phi_n \to g$ in $\mathscr{D}'$. Thus g represents the distribution f.

To see that $T^k_a f$ is actually the Taylor polynomial of g at a, we argue inductively. It suffices to give the argument for the identity

$$\frac{\partial g}{\partial x_1} = \frac{\partial}{\partial x_1} T^k_a f \ \text{ at } a.$$

If B is a ball about a, then

$$\phi_n \to g$$

$$\frac{\partial \phi_n}{\partial x_1} \to h$$

uniformly on B, where $h(b) = \dfrac{\partial}{\partial x_1}(T^k_b)(b)$. An elementary argument then yields that

$$\frac{\partial g}{\partial x_1} = h$$

inside B, which yields the desired result on evaluation at a.

That's it. ■

3. Examples

We illustrate the theorem by working a few examples. The only novelty here is the point of view: all the results are known (see, for instance, Triebel 1983).

Example 1. VMO.

Consider the Sarason space VMO, of functions of vanishing mean oscillation. A function $f \in L^1_{loc}(\mathbb{R}^d)$ belongs to VMO provided that

$$\sup_{\substack{Q \text{ a cube} \\ 0 < |Q| \leqslant t}} MO(f, Q) \to 0,$$

as $t \downarrow 0$, where

$$MO(f, Q) = \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(x) - c| dx$$

denotes the mean oscillation of f on Q. A norm on VMO is

$$f \mapsto \int_{B(0,1)} |f(x)| dx + \sup_{0 < |Q| < 1} MO(f, Q).$$

With this norm, VMO is an SCBS, and $\mathscr{D}$ is dense in VMO.

Now VMO has order $-\infty$, i.e. it does not inject locally into C^0. The natural way to see this is to write down some function f with $f \in \text{VMO} \sim C^0$. But in the spirit of the present discussion, we show it by proving that $\theta_0(r) \to 0$ as $r \downarrow 0$, i.e. there exist functions $\phi \in \text{VMO}$ such that $\phi = 1$ near 0, yet the VMO-norm $\|\phi\|$ is near 0.

Take $b > 0$, small, and let $\phi \in L^1_{cs}$ be defined by

$$\phi(x) = \max\{0, \min\{1, 1 + b - |x|^b\}\}$$

for $x \in \mathbb{R}^d$. Then $\phi = 1$ near 0 and the VMO norm of ϕ is bounded by a constant times b.

Example 2. Sobolev spaces.

For $0 \leqslant k \in \mathbb{Z}$ and $1 \leqslant p \leqslant +\infty$, the Sobolev space $W^{k,p}$ consists of those $f \in L^p$ such that all (distributional) partial derivatives $\partial^i f$ of order $|i| \leqslant k$ are representable by integration against L^p functions. With the norm

$$f \mapsto \|f\|_{L^p} + \| |D^k f| \|_{L^p},$$

the space $W^{k,p}$ becomes an SCBS. The subspace $\mathscr{D}$ is dense, provided $p < +\infty$. Sobolev's theorem says that $W^{k,p}$ has order greater than r if $kp > rp + d$, and has order less than r if $kp < rp + d$. The cases when $kp = rp + d$ vary. We consider a few cases.

Case 1. $kp < d$.

There is a constant $\kappa > 0$, depending on d and k, such that for each $r > 0$, there is a function $\phi_r \in \mathscr{D}$ such that $\phi_r = 1$ on $\mathbb{B}(0, r)$, $\phi_r = 0$ off $\mathbb{B}(0, 2r)$, and

$$|x|^j \cdot |D^j \phi_r(x)| \leqslant \kappa$$

whenever $0 \leqslant j \leqslant k$. Thus a calculation shows

$$\theta_0(r) \leqslant \| \phi_r \|_{W^{k,p}} \leqslant \kappa \cdot r^{d - kp/p}$$

(with a new $\kappa = \kappa(d, k, p)$). Thus the order of $W^{k,p}$ is $-\infty$.

Case 2. $p > d$, *and* $p < +\infty$.

We will show that the order of $W^{1,p}$ is 0.

First, consideration of the functions $x_1 \cdot \phi_r$, where ϕ_r is chosen as in the first case, yields

$$\theta_1(r) \leqslant \| x_1 \phi_r \|_{W^{1,p}} \leqslant \kappa \cdot r^{d/p},$$

with κ independent of r. Thus the order is less than 1.

The other direction is more trouble. This is typical. We have to show that if a function $\phi \in \mathscr{D}$ is identically 1 near 0, then it cannot have very small norm.

Suppose that ϕ is such a function. Suppose that $\| \phi \|_{L^p} < \frac{1}{4}$. Then there is a set $E \subset (\mathbb{B}(0, 2) \sim \mathbb{B}(0, 1))$ having volume at least 1, on which $|\phi| < \frac{1}{4}$. For each $a \in E$, the line integral of $|D\phi|$ on the ray $[0, a]$ from 0 to a exceeds $\frac{3}{4}$. Thus

$$\frac{3}{4} \leqslant \int_E \int_{[0,x]} |D\phi| ds\, dx.$$

$$\leqslant \kappa \cdot \int_{\mathbb{B}(0,2)} \frac{|D\phi(y)|}{|y|^{d-1}} dy$$

$$\leqslant \kappa \cdot \|D\phi\|_{L^p} \cdot \| y \mapsto \frac{1}{|y|^{d-1}} \|_{L^{p'}(\mathbb{B}(0,2))}.$$

where p' is the conjugate index to p. The condition $p > d$ guarantees that

$$\| \frac{1}{|y|^{d-1}} \|_{L^{p'}(\mathbb{B}(0,2))} < +\infty.$$

Thus, if $\| \phi \|_{L^p}$ is very small, then $\| D\phi \|_{L^p}$ is not, so in either case $\| \phi \|_{W^{1,p}}$ is not very small. Thus $\theta_0 > 0$, as claimed.

As a final case, we consider one of the borderline cases.

Case 3. $W^{d,1}$.

As before, it is easy to show that the order is less than 1.

To see that it is exactly 0, suppose $\phi \in \mathscr{D}$ is 1 at 0. Then

$$1 = \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \frac{\partial^d f(x_1,\ldots,x_d)}{\partial x_1 \ldots \partial x_d}\, dx_1 \ldots dx_d.$$

From this it is evident that $\|\phi\|_{W^{d,1}}$ cannot be very small.

Example 3. Campanato spaces.

Consider the scale of generalised mean oscillation spaces on $\mathbb{R}^d$. These were introduced by Campanato and by Meyers, following preliminary work of Morrey.

Fix $s \in [0, \infty]$ (smoothness level), $p \in [1, \infty)$ (integrability level), and $q \in [1, \infty)$ (fine tuning level). The space BMO(s, p, q) may be defined as follows.

Fix k to be the least integer greater than or equal to s.

For $f \in L^1_{\text{loc}}$ and a closed cube $Q \subset \mathbb{R}^d$, we denote by $T_Q f$ the unique polynomial $p \in \mathbb{C}[x_1,\ldots,x_d]$, such that

$$\int_Q (f - p)\cdot x^i\, dx = 0,$$

whenever i is a multi-index of order $|i| \leqslant k$. We define the kth order mean oscillation of f on Q as

$$MO(f, k, Q) = \frac{1}{|Q|}\int_Q |f(x) - T_Q f(x)|\, dx.$$

For $t > 0$ and $x \in \mathbb{R}^d$, we set

$$\Omega(f,x,t) = \sup_{\substack{|Q| = t^d \\ x \in Q}} MO(f, k, Q),$$

and then we set

$$\omega(f, t) = \|x \mapsto \Omega(f, x, t)\|_{L^p}.$$

Finally, we say that $f \in$ BMO(s, p, q) if the function $t \mapsto \omega(f, t)/t^s$ belongs to the space $L^q(dt/t)$. A norm on this space is

$$f \mapsto \|f\|_{L^1(B(0,1))} + \left(\int_0^\infty \left(\frac{\omega(f, t)}{t^s}\right)^q \frac{dt}{t}\right)^{1/q}.$$

With this norm, it is an SCBS, as may be seen.

Roughly speaking, the space BMO(s, p, q) is very close to the Sobolev space $W^{s,p}$ when s is integral, and for non-integral s interpolates between the integral values. The order of BMO(s, p, q) is $-\infty$ when $sp < d$, is 0 when $d < sp < d + p$, is 1 when $d + p < sp < d + 2p$, and so on (regardless of the value of q). The cases when one of these inequalities is replaced by equality are more delicate.

Consider the case $sp < d$. Estimation shows that the coefficient of x^i in the polynomial $T_Q f$ is bounded by $\kappa(d, k)$ times

$$(\text{side } Q)^{-|i|} \cdot \int_Q |f(y)|\, dy,$$

and thus

$$\int_Q |T_Q f|\, dx \leqslant \kappa \int_Q |f|\, dx.$$

Using this, we obtain that

$$MO(f, k, Q) \leqslant \kappa \inf_{p \in \mathbb{C}[x]_k} \frac{1}{|Q|} \int_Q |f(x) - p(x)|\, dx .$$

Given this observation, it is not too hard to use the same functions ϕ_r of Example 1 to show that $\theta_0(r) \downarrow 0$ for BMO(s, p, q). In fact, one obtains the following: (1) for x belonging to the support of ϕ_r and for small positive t,

$$\Omega(\phi_r, x, t) \leqslant \kappa \cdot \left(\frac{t}{r}\right)^{k+1};$$

(2) for dist(x, sptϕ) $> t$,

$$\Omega(\phi_r, x, t) = 0;$$

(3) for large t,

$$\Omega(\phi_r, x, t) \leqslant \kappa \cdot \left(\frac{r}{t}\right)^d.$$

Thus

$$\omega(\phi_r, t) \leqslant \kappa \cdot (r + t)^{d/p} \cdot \min\{(t/r)^{k+1}, (r/t)^d\},$$

and a calculation shows that the norm of ϕ_r is essentially $r^{d/p-s}$.

It seems to be a good deal harder to prove that the order is at least 0 when $sp > d$. I did not manage to find a proof that was shorter than the route via Campanato's identification of the space BMO(s, p, q) with the corresponding Besov space, and a classical proof of the embedding for the Besov space. To demonstrate a fairly general positive imbedding theorem, we close with the following example.

Example 4. The classical Besov spaces.

Let s, p and q be as in Example 3. Let k denote the least integer greater than s.

For $f \in L^p_{\text{loc}}$ and $t > 0$, consider the kth order L^p modulus of continuity

$$\omega(t) = \sup_{\substack{|h| \leqslant t \\ h \in \mathbb{R}^d}} \|x \mapsto \Delta_h^k f(x)\|_{L^p}.$$

Here, Δ_h^k denotes the kth power of the difference operator

$$\Delta_h : g(\cdot) \mapsto g(\cdot + h).$$

We say that f belongs to the Besov space BES(s, p, q) (or $B^s_{p,q}$) if the function

$$t \mapsto \frac{\omega(t)}{t^s}$$

belongs to $L^q(dt/t)$. A norm on BES(s, p, q) is

$$f \mapsto \|f\|_{L^p(\mathbb{B}(0,1))} + \left\{\int_0^\infty \left[\frac{\omega(t)}{t^s}\right]^q \frac{dt}{t}\right\}^{1/q}.$$

With this norm, BES(s, p, q) becomes an SCBS. The subspace $\mathscr{D}$ is dense except when p or q is $+\infty$.

Except for some of the cases when s is integral, or p or q equal 1 or $+\infty$, the Besov spaces BES(s, p, q) are in fact locally the same as the BMO(s, p, q), although this is far from obvious.

In particular, order(BES(s, p, q)) $\geqslant 0$ whenever $sp > d$. We will not prove this here, but will prove the weaker statement that the order of BES(s, p, ∞) is $\geqslant 0$ whenever

$$s > \frac{d}{p} + 1 - \frac{1}{p}.$$

Suppose $s > \frac{d}{p} + 1 - \frac{1}{p}$, and $\phi \in \mathscr{D}$ is identically 1 near 0. We have to show that its BES(s, p, ∞)-norm is bounded away from zero.

We may suppose that ϕ has small norm in $L^p(\mathbb{B}(0, 1))$, say norm at most one quarter the norm of the constant 1. Then for each small $t > 0$, there exists a square S of side $2t$, contained in $\mathbb{B}(0, 1)$, with

$$\left(\int_S |\phi|^p dx\right)^{1/p} \leqslant \frac{t^{d/p}}{4}.$$

We can then construct a chain of touching congruent squares, $S_1, S_2, \ldots, S_m$, each of side t, with S_1 centred at 0, S_m lying inside S, and $S_{j+1} = S_j + h$, where $|h| = t$. Note that $m \leqslant 1/t$.

Provided t is small enough, we will have $\phi = 1$ on S_1, so we obtain the estimate

$$\frac{3}{4}\cdot t^{d/p} < \left\{\int_{S_1} |\phi(x)|^p dx\right\}^{1/p} - \left\{\int_{S_m} |\phi(x)|^p dx\right\}^{1/p}$$

$$\leqslant \left\{\int_{S_1} |\phi(x) - \phi(x+mh)|^p dx\right\}^{1/p}$$

$$\leqslant \sum_{1}^{m-1} \left\{\int_{S_i} |\phi(x) - \phi(x+h)|^p dx\right\}^{1/p}$$

$$\leqslant t^{-1/p'} \cdot \| |\Delta_h \phi| \|_{L^p},$$

where p' is the conjugate index to p.

In rather similar fashion, we can obtain the higher-order estimate

$$\kappa \cdot t^{d/p} \leqslant t^{-1/p'} \cdot \| |\Delta_h^k \phi| \|_{L^p},$$

corresponding to the integer k. (For odd k we represent $\phi(x) - \phi(x+mh)$ as a sum of kth order differences, with a small error. For even k, we use an alternating sum of kth order differences, instead.) Thus, for all small $t > 0$, we obtain

$$\omega(\phi, t) \geqslant \kappa \cdot t^{d/p+1-1/p},$$

whence the BES(s, p, ∞)-norm of ϕ exceeds κ.

REFERENCE

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