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Uniform approximation by polynomials in two functions

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1. Introduction

In [7] we announced the following result:

Let $0 < k \in \mathbf{Z}$ *and let* f *be a* C^k *diffeomorphism of* $\mathbf{C}$ *into* $\mathbf{C}$, *reversing orientation. Then* $\mathbf{C}[z, f]$ *is dense in* $C^k(\mathbf{C}, \mathbf{C})$.

As a corollary, $\mathbf{C}[z, f]$ is dense in $C^0(\mathbf{C}, \mathbf{C})$, i.e. $\mathbf{C}[z, f]$ is dense in $C(X)$ for each compact X in $\mathbf{C}$. This verifies an old conjecture of A. Browder. We noted in [7] that the corollary also holds for direction-reversing *homeomorphisms* f, provided f is locally C^1 and nonsingular off a reasonably small closed set $E \subset \mathbf{C}$. The purpose of this paper is to present and prove a more general result of this kind, covering functions $f: \mathbf{C} \to \mathbf{C}$ that are merely finite-to-one (instead of one-to-one) off the exceptional set, and almost unrestricted on the exceptional set. Specifically, we consider *proper* functions that are r-fold smooth covering maps off a closed exceptional set E which has no interior and for which $\mathbf{C} \sim E$ has no bounded components. By a proper function, we mean a continuous function for which the pre-image of each compact set is compact.

Here is our result.

Theorem. *Suppose* $f: \mathbf{C} \to \mathbf{C}$ *proper. Suppose* $E \subset \mathbf{C}$ *is closed,* $\operatorname{int} E = \emptyset$, *and* $\mathbf{C} \sim E$ *has no bounded component. Suppose* $0 < r \in \mathbf{Z}$, *and for each* $a \in \mathbf{C} \sim E$ *we have* $f^{-1}f(a) \subset \mathbf{C} \sim E$ *and* $\# f^{-1}f(a) = r$. *Suppose that on* $\mathbf{C} \sim E$, f *is* C^1, *nonsingular, and locally direction-reversing. Then* $\mathbf{C}[z, f]$ *is dense in* $C^0(\mathbf{C}, \mathbf{C})$.

Observe that the hypotheses imply that $f(E)$ is disjoint from $f(\mathbf{C} \sim E)$, and that the exceptional set E embraces all the following: (a) the points $a \in \mathbf{C}$ such that the level set $f^{-1}f(a)$ is infinite, or is finite but has anything other than r points, (b) the points a at which f is not differentiable, and (c) the points a at which f is differentiable and critical.

The possibility of such a result is suggested by the observation that if $p(z) \in \mathbf{C}[z]$, nonconstant
then $\mathbf{C}[z, p(\bar{z})]$ is dense in $C^0(\mathbf{C}, \mathbf{C})$ (—a quick direct proof of this latter fact is

obtained by noting that $\mathbf{C}[z, p(\bar{z})]$ is a module over the self-adjoint algebra $\mathbf{C}[p(z), p(\bar{z})]$, and hence each extreme norm 1 annihilating measure for $\mathbf{C}[z, p(\bar{z})]$ is supported on some level set of $p(z)$. This method uses an idea familiar from the work of Bishop, and one previously used by de Branges in his celebrated proof of the Stone–Weierstrass theorem).

A less trivial consequence of the theorem is that if ϕ and ψ are C^1 diffeomorphisms of $\mathbf{C}$ onto $\mathbf{C}$, having opposite degrees, and if $p(z) \in \mathbf{C}[z]$, then $\mathbf{C}[\phi, p(\psi)]$ is dense in $C^0(\mathbf{C}, \mathbf{C})$. To deduce this, first make a change of variables to reduce to the case $\phi(z) = z$, and then apply the theorem with E equal to the (finite) set $\psi^{-1}(p^{-1}p(p')^{-1}(0))$ (= the preimage of the image of the set of zeros of p').

We remark that the main difficulty in proving theorems like this is to establish the *polynomial convexity* of the sets

$$X_R = \{(z, f(z)) : |z| \leq R\}.$$

In fact, if we were willing to assume that each X_R is polynomially-convex, then we could drop most of the assumptions in the statement of the theorem. See [10].

The hypotheses that f be C^1, nonsingular and locally direction-reversing on $\mathbf{C} \sim E$ may be re-expressed: f is C^1 and $|f_{\bar{z}}| > |f_z|$ on $\mathbf{C} \sim E$.

The theorem from [7], quoted above, follows from the case $r = 1$ of the present theorem. To see this, note first that it is sufficient to prove that $\mathbf{C}[z, f]$ is dense in $C^k(B)$ for each closed ball B. Fixing B, we may modify f off a neighbourhood of B to obtain a diffeomorphism f_1 of $\mathbf{C}$ *onto* $\mathbf{C}$. Then the hypotheses of the Theorem are satisfied, with $f = f_1$, $E = \varnothing$ and $r = 1$. Thus $\mathbf{C}[z, f]$ is dense in $C^0(B)$, hence the graph of $f|B$ is polynomially-convex. Now apply the functional calculus for Banach algebras and the Range-Siu theorem.

2. Proof of Theorem

For $a \in \mathbf{C} \sim E$, we denote the solutions of $f(z) = f(a)$ by $a_1, \ldots, a_r$, with the convention $a_1 = a$. Similarly, $f^{-1}f(b) = \{b_1, \ldots, b_r\}$, and so on. By hypothesis, the a_j $(1 \leq j \leq r)$ are distinct and lie in $\mathbf{C} \sim E$. Let

$$A_i(a) = \prod_{j \neq i} f_{\bar{z}}(a_j), \quad B_i(a) = \prod_{j \neq i} (a_i - a_j)^{-1}.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_r)$, where α_i is a value of $\log(A_i B_i)$, and let β be any complex number. Consider the polynomial in z:

$$p(z) = p(a, \alpha, \beta, z) = \sum_{i=1}^{r} \prod_{j \neq i} \left(\frac{z - a_j}{a_i - a_j} \right) \alpha_i + \beta \prod_{i=1}^{r} (z - a_i).$$

Evidently, $\exp p(a_i) = A_i B_i$ $(1 \leq i \leq r)$.

Suppose $\Gamma \subset \mathbf{C} \sim E$ is an arc, with endpoints b and c. We wish to choose $\alpha_i(a)$ and $\beta(a)$, for $a \in \Gamma$, so that $p(z)$ becomes $p(a, z)$, a *continuous* function of a on Γ, with values in $\mathbf{C}[z]$.

Observe first that we may choose $a_2, \ldots, a_r$ to depend continuously on a, for $a \in \Gamma$. Indeed, by an application of the inverse function theorem, $f^{-1}f(\Gamma)$ is a covering space of Γ, and hence consists of r pairwise-disjoint arcs, so that $a_2, \ldots, a_r$

are uniquely-determined (as continuous functions) by their initial values $b_2, \ldots, b_r$ at the initial point of Γ. The functions A_i and B_i now become continuous functions of a on Γ, and they are bounded, and bounded away from zero, on Γ. There is a unique continuous choice of $\alpha_i(a)$, once $\alpha_i(b)$ is specified. The function $a \mapsto \beta(a) \in C(\Gamma)$ may be chosen *at will*. With these choices made, $p(a, z)$ is continuous in a on Γ, as required.

Fix $R > 0$. We will show that $\mathbf{C}[z, f]$ is dense in $C(D)$, where

$$D = B(R) \overset{\text{def}}{=} \{z \in \mathbf{C} : |z| \leqq R\}.$$

This will suffice.

Let A denote the uniform algebra $\operatorname{clos}_{C(D)} \mathbf{C}[z, f]$.

Abbreviate $X = X_R = \{(z, f(z)) \in \mathbf{C}^2 : z \in D\}$.

Let $P(X)$ denote the uniform algebra $\operatorname{clos}_{C(X)} \mathbf{C}[z, w]$.

Clearly the continuous algebra homomorphism sending $z \mapsto z$ and $w \mapsto f(z)$ is an isomorphism from $P(X)$ onto A.

Fix $b \in D \sim E$.

Let $R_1 = 2 \cdot \max\{2^r \cdot R, \sup_D |f|\}$.

Choose $c \in \mathbf{C} \sim E$ such that c belongs to the same component of $\mathbf{C} \sim E$ as b and such that

$$|c_j| > R_1 \quad (1 \leqq j \leqq r),$$
$$|f(c)| > 2R_1.$$

This is possible, because f is continuous and proper, so that the sets $f^{-1} f B(R_1)$ and $f^{-1} B(2R_1)$ are compact, and because no component of $\mathbf{C} \sim E$ is bounded. Next, take an arc Γ from b to c in $\mathbf{C} \sim E$. Choose initial values $b_i(b), \alpha_i(b)$. Then a_i and α_i are determined on Γ. Finally, pick $\beta \in C(\Gamma)$ so that $\Re p(c, z) \geqq 0$ for $|z| \leqq R$ (Here, $\Re$ denotes the real part). This is a restriction only on the value $\beta(c)$. It can be met because $\prod_{i=1}^{r} (z - c_j)$ stays in an acute-angular sector, for $|z| \leqq R$.

Define $g(a, z) = \exp p(a, z)$. Then for $a \in \Gamma$, $g(a, z)$ is an entire function of z, and we have

$$g(a, z) = A_i B_i (1 + \tau_i(z))$$

where $\tau_i(z) = \tau_i(a, z) \to 0$ as $z \to a_i$. This convergence is *uniform* in $a \in \Gamma$, in the sense that, given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$\left. \begin{array}{l} a \in \Gamma \\ |z - a_i| < \delta_1 \end{array} \right\} \Rightarrow |\tau_i(a, z)| < \varepsilon.$$

This is so, because all the functions $f_z(a_i)^{\pm 1}, (a_i - a_j)^{\pm 1}$ for $i \neq j$, $A_i^{\pm 1}, B_i^{\pm 1}, \alpha_i$, and β are bounded for $a \in \Gamma$.

Define

$$\phi(a, z) = g(a, z)\{f(z) - f(a)\} \prod_{j=1}^{r} \left\{ \frac{z - a_j}{f_z(a_j)} \right\},$$

$$\psi(a, z) = g(a, z)\{f(z) - f(a)\} \prod_{j=2}^{r} \left\{ \frac{z - a_j}{f_z(a_j)} \right\}.$$

We may write

$$f(z)-f(a)=f_z(a_i)(z-a_i)+f_{\bar z}(a_i)(\bar z-\bar a_i)+\eta_i(a,z)|z-a_i|$$

where $\eta_i\to 0$ as $z\to a_i$. Again, the convergence is uniform in $a\in\Gamma$, because of the continuity of the derivative Df on $C\sim E$.

The hypothesis that f is locally direction-reversing now yields

$$(z-a_i)(f(z)-f(a))=f_{\bar z}(a_i)|z-a_i|^2\{1+\xi_i(a,z)\}$$

where $\limsup_{z\to a_i}|\xi_i|\leqq\kappa<1$. Here κ may be chosen independently of $a\in\Gamma$, and the convergence is uniform in $a\in\Gamma$. In addition to the above-mentioned facts and estimates, this depends on the fact that the ratio $|f_z(a_i)/f_{\bar z}(a_i)|$ is bounded below 1, uniformly on Γ.

Next, we may write

$$\prod_{j\neq i}(z-a_j)=\prod_{j\neq i}(a_i-a_j)\{1+\gamma_i(a,z)\}$$

where $\gamma_i\to 0$ as $z\to a_i$, uniformly in $a\in\Gamma$. Thus

$$\phi(a,z)=|z-a_i|^2\{1+\zeta_i(a,z)\}$$

where $\limsup_{z\to a_i}|\zeta_i|<1$, uniformly in $a\in\Gamma$. Consequently, there exists $\delta>0$ such that

$$a\in\Gamma,\quad |z-a_i|<\delta\Rightarrow\Re\phi(a,z)\geqq 0.$$

But, on the set $\{z\in\mathbf{C}:|z-a_i|\geqq\delta,\forall i\}$, the function $\phi(a,z)$ never vanishes. By compactness, $\phi(a,z)$ is bounded away from 0 on the set

$$\Gamma\times(D\cap\{z\in\mathbf{C}:|z-a_i|\geqq\delta,\forall i\}).$$

Thus, for $a\in\Gamma$ and $z\in D$, $\phi(a,z)$ omits the half-disc

$$\{z\in\mathbf{C}:\Re z<0,|z|<\lambda_1\},$$

for some $\lambda_1>0$. In particular,

$$\phi(a,z)+\lambda\neq 0\quad(a\in\Gamma,z\in D,0<\lambda<\lambda_1).$$

Define

$$\chi(a,\lambda,z,w)=g(a,z)(w-f(a))\prod_{j=1}^{r}(z-a_j)+\lambda\prod_{j=1}^{r}f_{\bar z}(a_j),$$

$$\lambda_2=\inf\left\{\frac{R^{r+1}}{|f_{\bar z}(a_i)|^r}:a\in\Gamma,1\leqq i\leqq r\right\}.$$

Then $\lambda_2>0$.

Let $|z|\leqq R,|w|\leqq R_1$, $0<\lambda<\lambda_2$. Then

$$|g(c,z)|=\exp\Re p(c,z)\geqq 1,$$

$$\left|g(c,z)(w-f(c))\prod_{j=1}^{r}(z-c_j)\right|>R^{r+1},$$

$$\chi(c,\lambda,z,w)\neq 0.$$

Thus $\chi(a, \lambda, z, w)$ is an entire function of (z, w), and for $0 < \lambda < \lambda_3 = \min\{\lambda_1, \lambda_2\}$, we have

(1) $\chi(a, \lambda, z, w) \neq 0$ if $(z, w) \in X$ and $a \in \Gamma$.

(2) $\chi(c, \lambda, z, w) \neq 0$ if $(z, w) \in \hat{X}$, the polynomially-convex hull of X.

Statement (2) holds, because the hull of X is certainly contained in the polydisc $B(R) \times B(R_1)$.

A classical argument of Oka now shows that $(z, w) \mapsto \chi(b, \lambda, z, w)^{-1}$ belongs to $P(X)$, whence $z \mapsto \phi(b, z) + \lambda \in A^{-1}$, for $0 < \lambda < \lambda_3$. For the reader's convenience, we give Oka's simple argument:

The point is that we have a function $h(a, z, w)$ which is continuous on $\Gamma \times \mathbf{C}^2$, which for each $a \in \Gamma$ is entire in (z, w) and nonvanishing on X, and which for $a = c$ belongs to $P(X)^{-1}$. Since the set of invertible elements in a Banach algebra is open, the set Σ of $a \in \Gamma$ for which $h(a, z, w) \in P(X)^{-1}$ is relatively open. Also, Σ is nonempty. If $\Sigma \neq \Gamma$ were possible, then without loss in generality we may assume $\Sigma = \Gamma \sim \{b\}$. Then $h(b, z, w)$ has a zero at some $(z_0, w_0) \in \hat{X} \sim X$. By continuity, $|h(a, z_0, w_0)^{-1}|$ takes on arbitrarily large values for a near b on Γ, but by the maximum principle, $|h(a, z_0, w_0)^{-1}|$ is bounded by $\sup_{(z,w) \in X} |h(a, z, w)^{-1}|$, which is bounded independently of a. Thus we have a contradiction, whence $\Sigma = \Gamma$.

So far, we have proved the following. Given $b \in D \sim E$, there exists a function $z \mapsto \phi(b, z) \in A$ and a $\lambda_3 > 0$ such that

$$0 < \lambda < \lambda_3 \Rightarrow \phi(b, z) + \lambda \in A^{-1}.$$

Now we will use this fact to conclude that $A = C(D)$. This part of the argument incorporates a device pioneered by Wermer [10].

Suppose μ is a complex Radon measure on D, annihilating A. We wish to show that $\mu = 0$, because (by the Riesz representation theorem and the Hahn-Banach theorem) this makes $A = C(D)$. Consider the Cauchy transform, $\hat{\mu}$, defined by

$$\hat{\mu}(z) = \frac{1}{\pi} \int \frac{d\mu(\zeta)}{z - \zeta},$$

whenever $z \in \mathbf{C}$ is such that the Newtonian potential

$$\tilde{\mu}(z) = \int \frac{d|\mu|(\zeta)}{|z - \zeta|}$$

is finite. The Newtonian potential is finite a.e. with respect to area, and the function $\hat{\mu}$ is locally integrable. Considered as a distribution, it satisfies

$$\frac{\partial \hat{\mu}}{\partial \bar{z}} = \mu.$$

In particular, the support of μ is a subset of the support of $\hat{\mu}$.

For $b \in \mathbf{C} \sim D$, $\hat{\mu}(b) = 0$, because μ annihilates $\mathbf{C}[z]$.

For $b \in D \sim E$, consider

$$\chi_n(b, z) = \frac{\psi(b, z)}{\phi(b, z) + \frac{1}{n}}, \quad n = 1, 2, 3, \ldots.$$

For $n > \lambda_3^{-1}$, we have $\chi_n(b,z) \in A$. Also,

$$\chi_n(b,z) \to \frac{f_{\bar{z}}(b)}{z-b}$$

as $n \uparrow \infty$, for $z \neq b_1, b_2, \ldots, b_r$. Since $\Re\phi(b,z) \geqq 0$ for z near b, we get

$$|\chi_n(b,z)| = \left| \frac{\phi(b,z)}{\phi(b,z) + \frac{1}{n}} \right| \cdot \frac{|f_{\bar{z}}(b)|}{|z-b|} \leqq \frac{|f_{\bar{z}}(b)|}{|z-b|}$$

for z near $b, z \neq b$. Hence there exists $c(b) > 0$ such that

$$|\chi_n(b,z)| \leqq \frac{c(b)}{|z-b|}$$

for $z \in D$. Given this much, the dominated convergence theorem tells us that

$$f_{\bar{z}}(b)\hat{\mu}(b) = \lim_{n \uparrow \infty} \frac{1}{\pi} \int \chi_n(b,z) d\mu(z) = 0$$

whenever $\tilde{\mu}(b) < \infty$ and $|\mu|[f^{-1}f(b)] = 0$. Thus $\hat{\mu}(b) = 0$ a.e. on $D \sim E$, with respect to area measure. Thus μ is supported on $E \cap D$.

Now $E \cap D$ is a compact set with no interior and connected complement, so Lavrentieff's theorem tells us that $\mathbf{C}[z]$ is dense in $C(E \cap D)$. Since μ annihilates $\mathbf{C}[z]$, we conclude that $\mu = 0$.

The proof is complete.

3. Concluding remarks

Apart from the examples mentioned, the theorem covers such algebras as

$$\mathbf{C}[z, x^m - \imath y^m],$$

where m is any odd natural number. Each of these algebras is dense in $C^0(\mathbf{C}, \mathbf{C})$. Examples like

$$\mathbf{C}[z, x\bar{z}],$$
$$\mathbf{C}[z, (x^2 - y^2)xy\bar{z}],$$

are not literally covered, because the functions f involved are not proper. However their density in $C^0(\mathbf{C}, \mathbf{C})$ does follow from the theorem, because on each compact set the functions coincide with proper functions that satisfy the hypotheses. (Alternatively, the argument of the theorem may be applied directly to them.)

Another class of examples is obtained by starting with a direction-reversing linear homeomorphism $g(z) = az + b\bar{z}(|b| > |a|)$, and function $\phi: [0, \infty) \to [0, \infty)$ such that $t\phi(t)^2$ is increasing, and forming $f = g^k\phi(|g|^{2k})$. A significant feature of the theorem is that the behaviour of f on the exceptional set E is quite unrestricted, apart from the requirement that f be proper. Thus E may include not only critical points of f, but points at which f is not differentiable at all, and E may contain infinite level sets of f. In [7] we stated that the following holds:

Let f be a direction-reversing homeomorphism of $\mathbf{C}$ *into* $\mathbf{C}$, *which is locally* C^1 *and noncritical off a closed set E, having area zero and not separating the plane. Then* $\mathbf{C}[z,f]$ *is dense in* $C^0(\mathbf{C},\mathbf{C})$.

In fact, the case $r=1$ of the theorem yields a considerably stronger result. The area of E does not matter, as long as it has no interior. It may separate the plane, as long as its complement has no bounded components. The function f need only be injective *off* E.

As a simple example, take E to be the segment $[-1,1]$, and the function $f=\bar{z}.\ \mathrm{dist}\,(z,E)$.

An open question along these lines, proposed by de Paepe, is whether $\mathbf{C}[p(\phi),q(\psi)]$ is dense in $C^0(\mathbf{C},\mathbf{C})$ whenever ϕ and ψ are homeomorphisms of opposite degrees, $p(z)$ and $q(z)$ belong to $\mathbf{C}[z]$, and $z\mapsto(p(\phi(z)),q(\psi(z)))$ is injective. For instance: $\mathbf{C}[z^2,\bar{z}^2+\bar{z}^3]$. A much broader (and probably unanswerable) question, is to classify the polynomially-convex hulls of all those $X\subset\mathbf{C}^n$ that are homeomorphic to the closed disc. Even for the case when X is a real-analytic disc, our understanding of this is unsatisfactory. There are some substantial partial results on the 'local' polynomial hull of discs (see, for instance, [1, 2, 4, 5, 9]). The theorems of [6, 7] and the present paper are among the few genuinely global results for discs.

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