PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 110, Number 3, November 1990

EXISTENCE RESULTS FOR SOME INITIAL- AND BOUNDARY-VALUE PROBLEMS

ANTHONY G. O'FARRELL AND DONAL O'REGAN

(Communicated by Kenneth R. Meyer)

ABSTRACT. Existence results for a variety of initial- and boundary-value problems are presented. For the problems considered, we show that the existence question depends on properties of the zero set of the nonlinearity. The analysis throughout is based upon a nonlinear alternative of A. Granas and the use of a priori bounds.

1. INTRODUCTION

This paper shows how existence results for various initial- and boundary-value problems may be deduced from the location of the zeros of the nonlinearity. The paper is divided into two main sections. In the first section, second-order problems of the form

$$(1.1)\qquad \begin{cases} \psi(t)y'' = f(t, y, y'), & 0 < t < 1, \\ y(0) = y(1) = 0, \end{cases}$$

are examined where $\psi: [0, 1] \to [0, \infty)$ and $f: [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ are continuous with $\psi > 0$ on $(0, 1)$. By a solution to (1.1) we mean a function $y \in C^1[0, 1] \cap C^2(0, 1)$ which satisfies the differential equation and boundary conditions. Boundary-value problems of the form (1.1) have been extensively examined; see for example [1, 4, 5, and 6]. In all of these papers f satisfies a growth condition in y' (usually at most quadratic) for (t, y) in bounded sets. However, if we examine the following two problems, which have virtually identical growth rates as $|y'| \uparrow \infty$,

$$(1.2)\qquad \begin{cases} t^{1/2}y'' = 2(t+1)\{1 - (y')^2\}, & 0 < t < 1, \\ y(0) = y(1) = 0, \end{cases}$$

$$(1.3)\qquad \begin{cases} t^{1/2}y'' = 2(t+1)\{1 + (y')^2\}, & 0 < t < 1, \\ y(0) = y(1) = 0, \end{cases}$$

Received by the editors May 1, 1989 and, in revised form, August 28, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34B15.
Key words and phrases. Existence, nonlinear, boundary-value problems, initial-value problems.

then it may be shown (see §2 below) that (1.2) has a solution, but (1.3) does not. What is really essential here is not the growth condition in y' but rather the location of the zeros of the polynomials $1-p^2$ and $1+p^2$. This will be explained in §2. The results of this paper improve, complement, and extend existing theory found in [5], which examined problems of the form (1.1) with $\psi = 1$ and $f(t, y, y') \equiv f(y')$. We remark as well that the results in §2 have obvious extensions to higher-order boundary-value problems; we choose, however, to omit the details.

In the second main section, §3, we examine the existence of solutions and their domains of definition for first-order initial-value problems of the form

$$\begin{cases} \psi(t)y' = f(t, y), & t \in (0, \tau], \\ y(0) = 0, \end{cases} \tag{1.4}$$

where $\psi\colon [0, \tau] \to [0, \infty)$ and $f\colon [0, \tau] \times \mathbf{R} \to \mathbf{R}$ are continuous with $\psi > 0$ on $(0, \tau]$. Here, by a solution to (1.4), we mean a function $y \in C[0, \tau] \cap C^1(0, \tau]$ which satisfies the differential equation and the initial condition. In [2] and [7], when the nonlinearity f satisfied a certain growth condition in y, global existence of a solution to (1.4) on $[0, \tau]$ was obtained, and in addition, for a certain class of functions f, τ was shown to be maximal. However, also in this case, if we examine two problems with virtually identical growth rate in y

$$\begin{cases} t^{1/2}y' = 1 - y^3, & t \in (0, \tau], \\ y(0), \end{cases} \tag{1.5}$$

$$\begin{cases} t^{1/2}y' = 1 + y^3, & t \in (0, \tau], \\ y(0), \end{cases} \tag{1.6}$$

it can be shown (see §3 below) that (1.5) has a solution for all $\tau > 0$ (i.e., global existence in the future), whereas (1.6) has a solution if $\tau < \pi^2/27$ but not if $\tau \geq \pi^2/27$. Again, what is really essential here is the location of the zeros of $1-y^3$ and $1+y^3$. The results of this paper, together with [2] and [7], provide an easy way to recognize, just by looking at the differential equation, when (1.4) has a solution either on $[0, \infty)$ or on $[0, \tau]$ for $\tau < \infty$. In addition, in the latter case we obtain the maximal τ for a certain class of problems. It should be remarked here as well that all the results in §3 have corresponding theorems for problems of the form

$$\begin{cases} \psi(t)y' = f(t, y), & t \in [-\tau, 0), \\ y(0) = 0. \end{cases}$$

The existence results of this paper are based on a nonlinear alternative [3], described below, of A. Granas, which reduces the problem of showing the existence of a solution to (1.1) to the problem of finding a priori bounds, independent of λ, for y and its first derivative y', when y is a solution to

$$\begin{cases} \psi(t)y'' = \lambda f(t, y, y'), & 0 < t < 1, \\ y(0) = y(1) = 0, \end{cases} \tag{1.7}$$

with $0 < \lambda < 1$. Similarly, the existence of a solution to (1.4) reduces to obtaining a priori bounds, independent of λ, for solutions y to

$$\begin{cases} \psi(t)y' = \lambda f(t, y), & 0 < t \le \tau, \\ y(0) = 0, \end{cases} \tag{1.8}$$

with $0 < \lambda < 1$.

Theorem 1.1 (nonlinear alternative). *Assume that U is a relatively open subset of a convex set K in a Banach space E. Let $N: \overline{U} \to K$ be a compact map, and assume that $0 \in U$. Then either*

(i) *N has a fixed point in $\overline{U}$, or*
(ii) *there is a point $u \in \partial U$ and a number $\lambda \in (0, 1)$ such that $u = \lambda N u$.*

Here, by a compact map is meant a continuous function whose image has compact closure. We have immediately

Theorem 1.2. *Assume that $0 < \lambda < 1$, that $f: [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ and $\psi: [0, 1] \to [0, \infty)$ are continuous with $\psi > 0$ on $(0, 1)$ and $1/\psi$ integrable on $[0, 1]$. Suppose there is a constant M, independent of λ, such that*

$$|y|_1 \stackrel{\text{def}}{=} \max\{|y|_0, |y'|_0\} < M,$$

(where $|y|_0 = \sup_{[0,1]} |y(t)|$) for each solution y to (1.7), for each $\lambda \in (0, 1)$. Then (1.1) has at least one solution in $C^1[0, 1] \cap C^2(0, 1)$.

Proof. Solving (1.7) is equivalent to finding a $y \in B$ (where B denotes the set of continuous functions on $[0, 1]$ satisfying the Dirichlet boundary conditions) which satisfies

$$y'(t) - y'(0) = \lambda \int_0^t \frac{f(s, y(s), y'(s))}{\psi(s)} ds.$$

Let

$$C_0[0, 1] = \{u \in C[0, 1]: u(0) = 0\},$$
$$C_B^1[0, 1] = \{u \in C^1[0, 1]: u \in B\}.$$

Define operators $L: C_B^1[0, 1] \to C_0$ and $N_f: C_B^1[0, 1] \to C_0$ by setting

$$(Ly)(t) = y'(t) - y'(0)$$
$$(N_f y)(t) = \int_0^t \frac{f(s, y(s), y'(s))}{\psi(s)} ds.$$

L is bijective and, by the Bounded Inverse Theorem, L^{-1} is a bounded linear operator. Thus (1.7) is equivalent to the fixed-point problem

$$y = \lambda L^{-1} N_f y \stackrel{\text{def}}{=} \lambda N y.$$

The operator N maps $C_B^1[0, 1]$ into itself. Now, N_f is clearly continuous, and it is also completely continuous, by the Arzela–Ascoli Theorem. Thus N

has the same properties. Set

$$U = \{u \in C_B^1[0, 1]: |u| < M + 1\},$$
$$K = C_B^1[0, 1],$$
$$E = C^1[0, 1].$$

Then Theorem 1.1 applies, but by the choice of U possibility (ii) is ruled out, and we deduce that N has a fixed point, i.e., (1.1) has a solution. □

Similarly, we have:

Theorem 1.3. *Assume that $0 < \lambda < 1$ and that $f: [0, \tau] \times \mathbf{R} \to \mathbf{R}$, and $\psi: [0, \tau] \to [0, \infty)$ are continuous with $\psi > 0$ on $(0, \tau]$ and $1/\psi$ integrable on $[0, \tau]$. Suppose that there is a constant M, independent of λ, such that*

$$\sup_{[0, \tau]} |y(t)| < M$$

for each solution y to (1.8), for each $\lambda \in (0, 1)$. Then (1.4) has at least one solution in $C[0, \tau] \cap C^1(0, \tau]$.

2. Boundary-value problems

Our main goal in this section is to obtain existence results for problems of the form (1.1), and we begin by treating the case of separable variables, $f(t, y, y') = \phi(t)g(y')$; subsequently we will use a comparison to treat the more general case. Consider first the problem

$$(2.1) \qquad \begin{cases} \psi(t)y'' = \phi(t)g(y'), & 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases}$$

If $g(0) = 0$, then (2.1) has the solution $y \equiv 0$. Thus for the remainder of this section we assume that $g(0) \neq 0$. Then either $g(0) > 0$ or $g(0) < 0$, and since the analysis and results are similar in both cases, we will assume without loss in generality that $g(0) > 0$.

Theorem 2.1. *Assume that ϕ, $\psi: [0, 1] \to [0, \infty)$ and $g: \mathbf{R} \to \mathbf{R}$ are continuous, that ϕ, $\psi > 0$ on $(0, 1)$, and that ϕ/ψ is integrable on $[0, 1]$. Also, suppose that $g(0) > 0$ and that g has two zeros of opposite sign. Let $r_2 < 0 < r_1$ be, respectively, the greatest negative and smallest positive root of g, and suppose that*

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds \leq \int_0^{r_1} \frac{du}{g(u)},$$
$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds \leq \int_{r_2}^0 \frac{du}{g(u)}.$$

Then (2.1) has at least one solution.

Proof. To prove the existence of a solution, we will apply Theorem 1.2, so it remains to show that there is a constant M, independent of λ, such that $|y|_1 < M$ for each solution y to

$$\begin{cases} \psi(t)y'' = \lambda\phi(t)g(y'), & 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases} \tag{2.2}$$

with $0 < \lambda < 1$. First, there exists a least $\tau_\lambda \in (0, 1)$ with $y'(\tau_\lambda) = 0$, and so, since $g(0) > 0$, we have $y'' > 0$ in a neighborhood of τ_λ.

Suppose that y'' attains the value 0 on $[\tau_\lambda, 1)$, and let δ be the first point in this interval with $y''(\delta) = 0$. Then $y'(\delta) = r_1$.

Thus $y'' > 0$ and $y' > 0$ on (τ_λ, δ). Consequently, on (τ_λ, δ) we have

$$\frac{y''}{g(y')} = \lambda\frac{\phi}{\psi},$$

and integration from τ_λ to δ yields

$$\int_0^{r_1} \frac{du}{g(u)} = \lambda \int_{\tau_\lambda}^{\delta} \frac{\phi(t)}{\psi(t)} dt < \int_0^1 \frac{\phi(t)}{\psi(t)} dt,$$

a contradiction.

Thus $y'' > 0$ on $[\tau_\lambda, 1)$, and hence $0 \le y' < r_1$ on $[\tau_\lambda, 1)$.

Analogous reasoning shows that $r_2 < y' \le 0$ on $(0, \tau_\lambda]$. Hence $r_2 \le y'(t) \le r_1$ for $t \in [0, 1]$, and so $r_2 \le y(t) \le r_1$ for $t \in [0, 1]$. The constant $M = 2\max\{r_1, -r_2\}$ thus has the desired property, and the existence of a solution to (2.1) follows. □

Theorem 2.2. *Assume that* ϕ, $\psi\colon [0, 1] \to [0, \infty)$ *and* $g\colon \mathbf{R} \to \mathbf{R}$ *are continuous, that* ϕ, $\psi > 0$ *on* $(0, 1)$ *and that* ϕ/ψ *is integrable on* $[0, 1]$.

(i) *Suppose that* $g(0) > 0$, *and that* g *has a positive zero but no negative zero. Let* $r_1 > 0$ *be the smallest positive root of* g, *and suppose that*

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds \le \int_0^{r_1} \frac{du}{g(u)},$$
$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_{-\infty}^0 \frac{du}{g(u)}.$$

Then (2.1) *has at least one solution.*

(ii) *Suppose that* $g(0) > 0$, *and that* g *has a negative zero but no positive zero. Let* $r_2 < 0$ *be the greatest negative root of* g, *and suppose that*

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds \le \int_{r_2}^0 \frac{du}{g(u)},$$
$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_0^{\infty} \frac{du}{g(u)}.$$

Then (2.1) *has at least one solution.*

Proof. (i) Let y be a solution to (2.2). Just as in Theorem 2.1, there exists a least $\tau_\lambda \in (0, 1)$ with $y'(\tau_\lambda) = 0$ and $0 \le y'(t) \le r_1$ for $t \in [\tau_\lambda, 1]$. In addition, $y'' > 0$ and so $y' < 0$ on some interval to the left of τ_λ. If y'' attains the value 0 at a point ϵ to the left of τ_λ, then $\phi(\epsilon)g(y'(\epsilon)) = 0$, a contradiction since g has no negative zeros. Thus $y'' > 0$ on $(0, \tau_\lambda)$, so $y'(0) \le y'(t) \le 0$ for $t \in [0, \tau_\lambda]$. Consequently

$$\int_{y'(0)}^{0} \frac{du}{g(u)} = \lambda \int_0^{\tau_\lambda} \frac{\phi(s)}{\psi(s)} ds < \int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_{-\infty}^{0} \frac{du}{g(u)},$$

so there exists a constant M_1, independent of λ, such that $-M_1 \le y'(0)$. Thus $-M_1 \le y'(t) \le r_1$ for $t \in [0, 1]$, and the existence of a solution to (2.1) now follows from Theorem 1.2, as before. □

Theorem 2.3. *Assume that* $\phi,\ \psi: [0, 1] \to [0, \infty)$ *and* $g: \mathbf{R} \to \mathbf{R}$ *are continuous, that* $\phi,\ \psi > 0$ *on* $(0, 1)$, *and that* ϕ/ψ *is integrable on* $[0, 1]$. *Suppose that* $g(0) > 0$ *and that* g *has no real zeros.*

(i) *Suppose that*

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_0^\infty \frac{du}{g(u)},$$
$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_{-\infty}^0 \frac{du}{g(u)}.$$

Then (2.1) *has at least one solution.*

(ii) *Suppose that*

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds \ge \int_{-\infty}^\infty \frac{du}{g(u)}.$$

Then (2.1) *does not have a solution.*

Proof. To prove (i), let y be a solution to (2.2). Now there exists a least $\tau_\lambda \in (0, 1)$ such that $y'(\tau_\lambda) = 0$ and, of course, $y'' > 0$ on $(0, 1)$. Integrating from τ_λ to any $t > \tau_\lambda$ yields

$$\int_0^{y'(t)} \frac{du}{g(u)} = \lambda \int_{\tau_\lambda}^{t} \frac{\phi(s)}{\psi(s)} ds < \int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_0^\infty \frac{du}{g(u)},$$

and similarly, integration from any $t < \tau_\lambda$ to τ_λ yields

$$\int_{y'(t)}^{0} \frac{du}{g(u)} < \int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_{-\infty}^{0} \frac{du}{g(u)}.$$

Thus there exists a constant M_1, independent of λ, such that $|y'(t)| \le M_1$ for $t \in [0, 1]$, so existence of a solution to (2.1) follows from Theorem 1.2.

To prove (ii), if y satisfies (2.1), then integration from 0 to 1 yields

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds = \int_{y'(0)}^{y'(1)} \frac{du}{g(u)} < \int_{-\infty}^{\infty} \frac{du}{g(u)},$$

which is impossible. □

We now turn our attention to the more general problem

$$(2.3)\qquad \begin{cases} \psi(t)y'' = f(t, y, y'), & 0 < t < 1, \\ y(0) = y(1) = 0, \end{cases}$$

and we will see that the above results may be generalized quite easily by using a comparison. Throughout the following analysis we will assume that

$$|f(t, y, p)| \le \phi(t)|g(p)|,$$

where ϕ and g will be described below. Once again, if $g(0) = 0$ then (2.3) has the solution $y \equiv 0$. Also, like cases $g(0) > 0$ and $g(0) < 0$ yield similar results to one another. Hence, without loss in generality we will assume that $g(0) > 0$.

Theorem 2.4. *Assume that*

- $f: [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ *and* $\psi: [0, 1] \to [0, \infty)$ *are continuous with* $\psi > 0$ *on* $(0, 1)$;
- *there are continuous functions* $\phi: [0, 1] \to [0, \infty)$ *and* $g: \mathbf{R} \to \mathbf{R}$ *with* $g(0) > 0$ *such that* $\phi > 0$ *on* $(0, 1)$, ϕ/ψ *is integrable on* $[0, 1]$, *and* $|f(t, y, p)| \le \phi(t)|g(p)|$; *and*
- $f(t, y, 0) > 0$ *for all* $t \in (0, 1)$ *and* $y \in \mathbf{R}$.

(i) *Suppose that*

- g *has two zeros of opposite sign, and*
- *if there exist* $r \in \mathbf{R}$, $t \in (0, 1)$, *and* $y \in \mathbf{R}$ *such that* $f(t, y, r) = 0$, *then* $g(r) = 0$.

Let $r_2 < 0 < r_1$ *be respectively the greatest negative and smallest positive roots of* g, *and suppose that*

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds \le \int_0^{r_1} \frac{du}{g(u)},$$
$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds \le \int_{r_2}^0 \frac{du}{g(u)}.$$

Then (2.3) *has at least one solution.*

(ii) *Suppose that*

- g *has a positive zero but no negative zero, and*
- *if there exist* $r > 0$, $t \in (0, 1)$, *and* $y \in \mathbf{R}$ *such that* $f(t, y, r) = 0$, *then* $g(r) = 0$.

Let $r_1 > 0$ *be the smallest positive root of* g, *and suppose that*

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds \le \int_0^{r_1} \frac{du}{g(u)},$$
$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_{-\infty}^0 \frac{du}{g(u)}.$$

Then (2.3) *has at least one solution.*

(iii) *Suppose that*

- g *has a negative zero but no positive zero, and*
- *if there exist* $r < 0$, $t \in (0, 1)$, *and* $y \in \mathbf{R}$ *such that* $f(t, y, r) = 0$, *then* $g(r) = 0$.

Let $r_2 < 0$ *be the greatest negative root of* g, *and suppose that*

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds \le \int_{r_2}^0 \frac{du}{g(u)},$$

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_0^\infty \frac{du}{g(u)}.$$

Then (2.3) *has at least one solution.*

(iv) *Suppose that*

- g *has no real zeros and*

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_0^\infty \frac{du}{g(u)},$$

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_{-\infty}^0 \frac{du}{g(u)}.$$

Then (2.3) *has at least one solution.*

Proof. To prove (i), let y be a solution to

$$\text{(2.4)} \qquad \begin{cases} \psi(t)y'' = \lambda f(t, y, y'), & 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases}$$

with $0 < \lambda < 1$. There exists $\tau_\lambda \in (0, 1)$ with $y'(\tau_\lambda) = 0$, so $y''(\tau_\lambda) > 0$ and consequently $y'' > 0$ in a neighborhood of τ_λ. Let δ be the least point (if any such exists) after τ_λ at which $y''(\delta) = 0$. Then $y'' > 0$ and $y' > 0$ on (τ_λ, δ), and $y'(\delta) = r_1$; thus, integration from τ_λ to δ yields

$$\int_0^{r_1} \frac{du}{g(u)} \le \lambda \int_{\tau_\lambda}^{\delta} \frac{\phi(t)}{\psi(t)} dt < \int_0^1 \frac{\phi(t)}{\psi(t)} dt,$$

a contradiction. Thus $y'' > 0$ on $(\tau_\lambda, 1)$ so $0 \le y'(t) \le r_1$ for $t \in [\tau_\lambda, 1]$. Analogous reasoning shows that $r_2 \le y'(t) \le 0$ for $t \in [0, \tau_\lambda]$, and existence of a solution to (2.3) now follows from Theorem 1.2.

The proofs of (ii) and (iii) are similar to the above and Theorem 2.2.

To prove (iv), let y be a solution to (2.4). Now there exists $\tau_\lambda \in (0, 1)$ with $y'(\tau_\lambda) = 0$ and so $y'' > 0$ on $(0, 1)$. Integration from τ_λ to any given $t > \tau_\lambda$ gives

$$\int_0^{y'(t)} \frac{du}{g(u)} < \int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_0^\infty \frac{du}{g(u)},$$

and similarly, integration from any $t < \tau_\lambda$ to τ_λ yields

$$\int_{y'(t)}^0 \frac{du}{g(u)} < \int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_{-\infty}^0 \frac{du}{g(u)}.$$

Existence of a solution now follows from Theorem 1.2. □

Remark. Many boundary conditions other than Dirichlet boundary conditions could be considered; however, since the analysis is similar in all cases we choose to omit the details. For example, if we were to consider

$$\text{(2.5)} \qquad \begin{cases} \psi(t)y'' = f(t, y, y'), & 0 < t < 1, \\ y'(0) = y(1) = 0, \end{cases}$$

then essentially the same reasoning yields the following parallel version of Theorem 2.4.

Theorem 2.4*. *Assume that*

- $f: [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ *and* $\psi: [0, 1] \to [0, \infty)$ *are continuous with* $\psi > 0$ *on* $(0, 1)$;
- *there are continuous functions* $\phi: [0, 1] \to [0, \infty)$ *and* $g: \mathbf{R} \to \mathbf{R}$ *with* $g(0) > 0$ *such that* $\phi > 0$ *on* $(0, 1)$ ϕ/ψ *is integrable on* $[0, 1]$, *and* $|f(t, y, p)| \le \phi(t)|g(p)|$;
- $f(0, y, 0) > 0$ *for all* $y \in \mathbf{R}$; *and*
- *if* $r > 0$ *is such that there exist* $t \in (0, 1)$ *and* $y \in \mathbf{R}$ *such that* $f(t, y, r) = 0$, *then* $g(r) = 0$.

(i) *Suppose that* g *has a positive zero. Let* $r_1 > 0$ *be the smallest positive root of* g, *and suppose that*

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds \le \int_0^{r_1} \frac{du}{g(u)}.$$

Then (2.5) *has at least one solution.*

(ii) *Suppose that* g *has no positive zeros and that*

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds < \int_0^{\infty} \frac{du}{g(u)}.$$

Then (2.5) *has at least one solution.* □

Example. The problem

$$\text{(2.6)} \qquad \begin{cases} t^{1/2}y'' = 2(t+1)\{1 - (y')^2\}, & 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases}$$

has a solution, whereas

$$\text{(2.7)} \qquad \begin{cases} t^{1/2}y'' = 2(t+1)\{1 + (y')^2\}, & 0 < t < 1, \\ y(0) = y(1) = 0 \end{cases}$$

does not. To see that (2.6) has a solution, take $\psi(t) = t^{1/2}$, $\phi(t) = 2(t+1)$, and $g(u) = 1 - u^2$. Note that g has two zeros of opposite signs, $r_1 = 1$ and $r_2 = -1$, and that $g(0) > 0$. In addition,

$$\int_0^{r_1} \frac{du}{g(u)} = \int_0^1 \frac{du}{1 - u^2} = \infty$$

and

$$\int_{r_2}^{0} \frac{du}{g(u)} = \infty .$$

Consequently all the conditions of Theorem 2.1 are satisfied, and hence (2.6) has a solution.

Now, to see that (2.7) has no solution, take $\psi(t) = t^{1/2}$, $\phi(t) = 2(t+1)$, and $g(u) = 1 + u^2$. Note that g has no real zeros, $g(0) > 0$, and

$$\int_0^1 \frac{\phi(s)}{\psi(s)} ds = \frac{16}{3} \geq \pi = \int_{-\infty}^{\infty} \frac{du}{g(u)} .$$

Thus (2.7) has no solution by Theorem 2.3.

Example. The problem

$$(2.8) \qquad \begin{cases} t^\alpha y'' = \frac{t^2+1}{y^2+1}(1-y')^m(2+y')^n(1+(y')^2), & 0 < t < 1 , \\ y(0) = y(1) = 0 \end{cases}$$

with $0 \leq \alpha < 1$, $n > 0$, $m > 0$, has a solution. To see this, take $\psi(t) = t^\alpha$, $\phi(t) = t^2 + 1$, and $g(u) = (1-u)^m(2+u)^n(1+u^2)$. Note that g has two zeros of opposite sign, $r_1 = 1$ and $r_2 = -2$, and that $g(0) > 0$. In addition,

$$\int_0^{r_1} \frac{du}{g(u)} = \int_{r_2}^{0} \frac{du}{g(u)} = \infty ,$$

and $f(t, y, 0) = (t^2+1)/(y^2+1) > 0$ for $t \in (0, 1)$ and $y \in \mathbf{R}$. Consequently, all the conditions of Theorem 2.4 are satisfied, and hence (2.8) has at least one solution.

Remark. As can be seen from the above analysis, there are obvious extensions to higher-order boundary-value problems. We omit the details.

3. Initial-value problems

Again in this section we treat first the case of separable variables and then generalize. Consider

$$(3.1) \qquad \begin{cases} \psi(t)y' = \phi(t)g(y), & t \in (0, \tau], \\ y(0) = 0. \end{cases}$$

We assume once again, without loss in generality, that $g(0) > 0$.

Theorem 3.1. *Assume ψ, ϕ: $[0, \tau] \to [0, \infty)$ and g: $\mathbf{R} \to \mathbf{R}$ are continuous with $\psi > 0$ on $(0, \tau]$, $\phi > 0$ on $(0, \tau)$, $g(0) > 0$, and ϕ/ψ integrable on $[0, \tau]$.*

(i) *Suppose that g has a positive zero. Let $r_1 > 0$ be the smallest positive root of g, and suppose that*

$$\int_0^{\tau} \frac{\phi(s)}{\psi(s)} ds \leq \int_0^{r_1} \frac{du}{g(u)} .$$

Then (3.1) *has at least one solution.*

(ii) *Suppose that* g *has no positive zero.*

(a) *If*

$$\int_0^\tau \frac{\phi(s)}{\psi(s)} ds < \int_0^\infty \frac{du}{g(u)},$$

then (3.1) *has at least one solution.*

(b) *On the other hand, if*

$$\int_0^\tau \frac{\phi(s)}{\psi(s)} ds \geq \int_0^\infty \frac{du}{g(u)},$$

Then (3.1) *has no solution.*

Proof. To prove the existence of a solution in (i), we apply Theorem 3.1, so it remains to show that there exists a constant M, independent of λ, such that $\sup_{[0,\tau]} |y(t)| < M$ for each solution y to

$$\text{(3.2)} \qquad \begin{cases} \psi(t)y' = \lambda\phi(t)g(y), & t \in (0, \tau], \\ y(0) = 0, \end{cases}$$

with $0 < \lambda < 1$. First, $y' > 0$ in a neighborhood of zero. Suppose that $y' > 0$ on $(0, \delta)$ and $y'(\delta) = 0$. Then $y(\delta) = r_1$. Consequently, integration from 0 to δ yields

$$\int_0^{r_1} \frac{du}{g(u)} = \lambda \int_0^\delta \frac{\phi(s)}{\psi(s)} ds < \int_0^\tau \frac{\phi(s)}{\psi(s)} ds,$$

a contradiction. Thus $y' > 0$ on $(0, \tau)$, so $0 < y(t) < r_1$ for $t \in (0, \tau)$. Now existence of a solution follows from Theorem 1.3.

To prove (ii) (a), let y be a solution to (3.2). Now $g(0) > 0$ implies $y' > 0$ and hence $y > 0$ on $(0, \tau)$, so integration from 0 to t yields

$$\int_0^{y(t)} \frac{du}{g(u)} = \lambda \int_0^t \frac{\phi(s)}{\psi(s)} ds < \int_0^\tau \frac{\phi(s)}{\psi(s)} ds < \int_0^\infty \frac{du}{g(u)}.$$

Existence of a solution to (3.1) now follows from Theorem 1.3.

To prove (b), let y be a solution to (3.1). Then integration from 0 to τ yields

$$\int_0^\tau \frac{\phi(s)}{\psi(s)} ds = \int_0^{y(\tau)} \frac{du}{g(u)} < \int_0^\infty \frac{du}{g(u)},$$

which is impossible. □

We now turn to the more general problem

$$\text{(3.3)} \qquad \begin{cases} \psi(t)y' = f(t, y), & t \in (0, \tau], \\ y(0) = 0. \end{cases}$$

Essentially the same reasoning as above establishes:

Theorem 3.2. *Assume that*

- $\psi: [0, \tau] \to [0, \infty)$ *and* $f: [0, \tau] \times \mathbf{R} \to \mathbf{R}$ *are continuous with* $\psi > 0$ *on* $(0, \tau]$;
- *there are continuous functions* $\phi: [0, \tau] \to [0, \infty)$ *and* $g: \mathbf{R} \to \mathbf{R}$ *with* $g(0) > 0$, $\phi > 0$ *on* $(0, \tau)$, ϕ/ψ *integrable on* $[0, \tau]$, *and* $|f(t, y)| \le \phi(t)|g(y)|$;
- $f(0, 0) > 0$; *and*
- *if* $r > 0$ *and* $f(t, r) = 0$ *for some* $t \in (0, \tau)$, *then* $g(r) = 0$.

(i) *Suppose that* g *has a positive zero. Let* $r_1 > 0$ *be the smallest positive root of* g, *and suppose that*

$$\int_0^\tau \frac{\phi(s)}{\psi(s)} ds \le \int_0^{r_1} \frac{du}{g(u)}.$$

Then (3.3) *has at least one solution.*

(ii) *Suppose that* g *has no positive zero. If*

$$\int_0^\tau \frac{\phi(s)}{\psi(s)} ds < \int_0^\infty \frac{du}{g(u)},$$

then (3.3) *has at least one solution.*

□

Example. The initial value problem

$$\text{(3.4)} \qquad \begin{cases} t^\alpha y' = (t^2+1)(1-y^n), & t \in (0, \tau], \\ y(0) = 0 \end{cases}$$

with $0 \le \alpha < 1$ and $n > 0$ has a solution. To see this, take $\psi(t) = t^\alpha$, $\phi(t) = t^2 + 1$, and $g(y) = 1 - y^n$. Note that g has a positive zero $r_1 = 1$ and that $g(0) > 0$. In addition,

$$\int_0^{r_1} \frac{du}{g(u)} = \infty,$$

and consequently (3.4) has a solution by Theorem 3.1. In fact (3.4) has a solution on $[0, \tau]$ for each $\tau > 0$.

Example. Now

$$\text{(3.5)} \qquad \begin{cases} t^{1/2} y' = 1 + y^3, & t \in (0, \tau], \\ y(0) = 0 \end{cases}$$

has a solution if $\tau < \pi^2/27$ and no solution if $\tau \ge \pi^2/27$. To see this, let $\psi(t) = t^{1/2}$, $\phi(t) = 1$, and $g(y) = 1 + y^3$. Note that g has no positive zeros and $g(0) > 0$. In addition,

$$\int_0^\tau \frac{\phi(s)}{\psi(s)} ds = 2\tau^{1/2} \qquad \text{and} \qquad \int_0^\infty \frac{du}{g(u)} = \frac{2\pi}{3\sqrt{3}}.$$

Hence Theorem 3.1 implies that (3.5) has a solution if $\tau < \pi^2/27$ and no solution if $\tau \ge \pi^2/27$.

References

1. L. E. Bobisud, D. O'Regan, and W. D. Royalty, *Singular boundary value problems*, Appl. Anal. **23** (1986), 233–243.
2. L. E. Bobisud and D. O'Regan, *Existence of solutions to some singular initial value problems*, J. Math. Anal. Appl. **133** (1988), 135–149.
3. J. Dugundji and A. Granas, *Fixed point theory*, vol. 1, Monographie Mathematyczne, PNN, Warsaw, 1982.
4. A. Granas, R. B. Guenther, and J. W. Lee, *Nonlinear boundary value problems for ordinary differential equations*, Dissertationes Math., Warsaw, 1985.
5. ——, *Nonlinear boundary value problems for some classes of ordinary differential equations*, Rocky Mountain J. Math. **10** (1980), 35–58.
6. ——, *Existence principles for classical and Carathéodory solutions for systems of ordinary differential equations*, Proc. Internat. Conf. on Theory and Applications of Differential Equations, Ohio University Press, Athens, Ohio, 1988, 353–364.
7. J. W. Lee and D. O'Regan, *Topological transversality* IV: *Applications to initial value problems*, Ann. Polon. Math. **48** (1988), 247–252.

Department of Mathematics, Maynooth College, Kildare County, Ireland