Some Problems about Area and Perimeter

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1 Introduction

We present a few problems about the area and perimeter of rectangles, suitable for consideration by secondary school students. They are arranged in approximate order of increasing difficulty, beginning with a problem that should be accessible to anyone who can solve a quadratic, and working up to some problems of Mathematical Olympiad standard. This material is offered for use in enrichment activities designed to stimulate talented students.

First, we list the problems. Then we offer a few hints, and then complete solutions. This format is designed to facilitate activity in which the student tries to work independently on the problems, before seeking help.

2 Problems

1. Suppose a rectangular yard has area $1200m^2$, and perimeter $160m$. What are its dimensions? (By its dimensions we mean its length and

breadth.)

2. Show that if one rectangle A contains a second rectangle B,

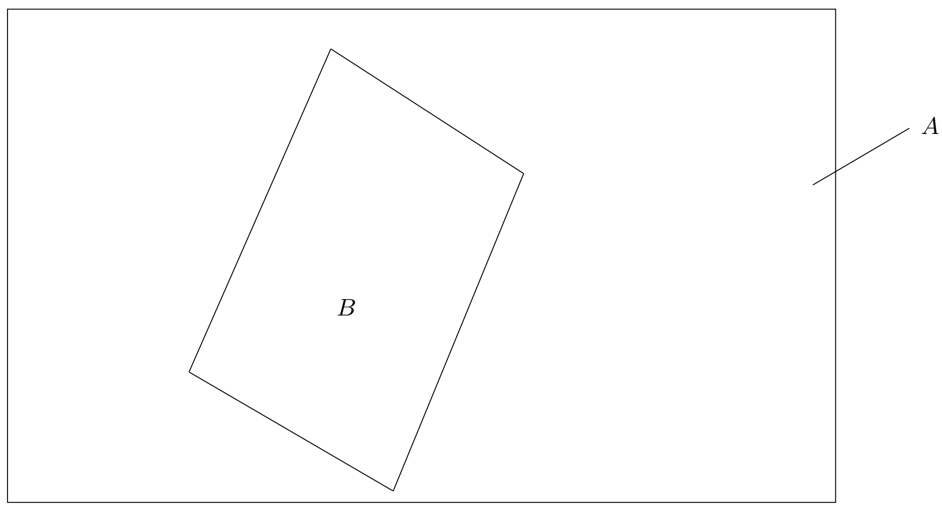

Figure 1: Rectangle B inside Rectangle A

then the perimeter of B is no longer than the perimeter of A.

3. Let us agree that a *patio* is a rectangle paved using tiles, and more than one tile wide. (If it were only one tile wide, it would be a path, not a patio!)
A patio is constructed, using square tiles of side $1m$. Its area is $91m^2$. What are its dimensions?

4. A patio is constructed using square tiles of side $1m$. Its area is $100m^2$.

   (a) How many possible shapes could it have? What are its possible dimensions?

   (b) Which shape has smallest perimeter?

   (c) Which has the largest?

5. A patio is constructed using square tiles of side $1m$. Its perimeter is $30m^2$.

   (a) How many possible shapes could it have? What are its possible dimensions?

   (b) Which shape has smallest area?

   (c) Which has the largest?

6. A patio is constructed using square tiles of side $1m$. Its area is $186m^2$, and its perimeter is less than $120m$. What is its shape?

7. Dad constructs a patio in the back garden, using square tiles of side $1m$. John and Mary are away in France, and have not yet seen it, but they know that the back garden is a rectangle with perimeter $150m$. John learns how many tiles were used, and someone tells Mary the length of the perimeter. John and Mary meet, and discuss the patio. The conversation goes as follows:

   Mary: I know the perimeter. I won't tell you what it is, but of course you know that it is less than $150m$, because that is the perimeter of the back garden.
   John: I can't tell what the dimensions are.
   Mary: I already knew that!
   John: In that case, I now know!
   Mary: Ah! Then I know, too!

   What are the dimensions of the patio?

8. The basic situation is the same as in the last problem, but the conversation is different:

   Mary: I know the perimeter. I won't tell you what it is, but it is less than $120m$.
   John: I can't tell what the dimensions are.
   Mary: I already knew that!
   John: Well, I still can't tell.
   Mary: In that case, I now know!

   What are the dimensions of the patio?

9. The basic situation is the same as in the last two problems, but the conversation is again different:

   Mary: I know the perimeter. I won't tell you what it is, but it is less than $120m$.
   John: I can't tell what the dimensions are.
   Mary: I already knew that!
   John: Well, I still can't tell.
   Mary: Neither can I.
   John: In that case, I now know!

   What are the dimensions of the patio?

In these problems, you should take it that the spaces between tiles are negligible. It is conceivable that the patios may be square, but otherwise you should take it that the length exceeds the breadth.

3 Hints

Let x metres be the length and y metres be the breadth of the rectangular yard or patio, in each case. So $x \geq y$. The area is then xy m^2 and the perimeter $2(x + y)$ m.

For Problem 1, write down two equations for x and y, eliminate one variable, and solve for the other.

For Problem 2, first consider the case in which the four corners of rectangle A lie on the edges of rectangle B.

The remaining problems involve some elementary facts about whole numbers, particularly prime numbers. It will be helpful to make a list of the first few primes, for reference. You can check that the primes less than 100 are: 2,3,5,7,11,13,17,19,23, 29,31,37,41,43,47,53,59,61,67,71,73,79, 83,89, and 97. (Notice that for a number n less than 100, to check that n is prime, you just have to check that it is not divisible by 2,3,5, or 7.)

The last three problems will take even a strong student some hours to think through, and require perseverence.

4 Solutions

4.1 Problem 1

We have the equations

$$\begin{cases} x + y = 80 \\ xy = 1200 \end{cases}$$

Eliminating y from the first and substituting in the second gives $y = 80 - x$ and the quadratic

$$x^2 - 80x + 1200 = 0$$

which yields two solutions $x = 60$, $y = 20$ or $x = 20$, $y = 60$. Since $x \geq y$, we must have $x = 60$ and $y = 20$.

4.2 Problem 2

Draw the smallest rectangle with sides parallel to A that contains B. Call this rectangle C. Each side of C is parallel to, and no longer than, one of the sides of A, so the perimeter of C is no longer than the perimeter of A.

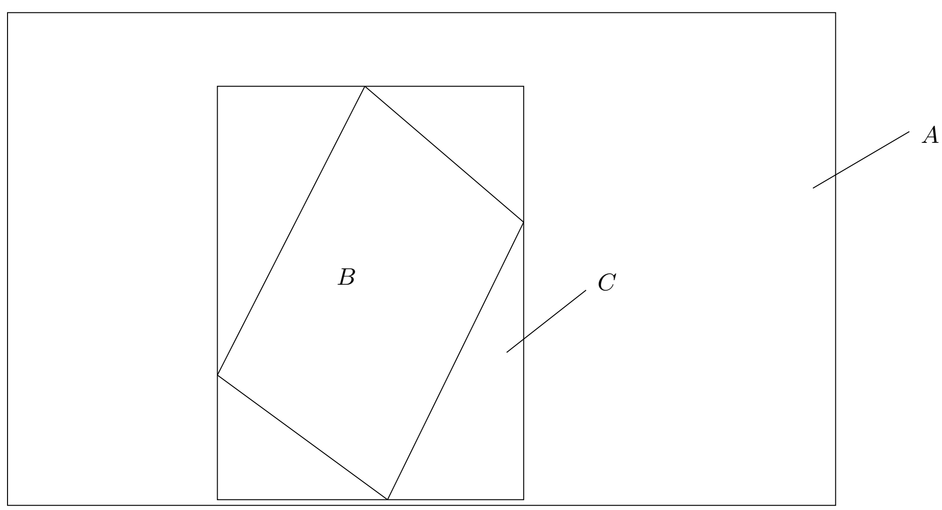

Figure 2: The Rectangles A, B and C

The corners of B lie on the edges of C.

If two corners lie on one edge of C, then $B = C$, and we are finished.

If B is not equal to C, then B meets the perimeter of C only at these corners, and the corners of B divide each edge of C in two segments. The 8 segments, together with the 4 edges of B, form 4 right-angle triangles. In each triangle, one edge of B is the hypotenuse, and hence is less than the sum of the other two sides (because two sides of a triangle are together greater than the third). Thus the perimeter of B is less than the perimeter of C, and hence less than that of A.

4.3 Problem 3

This time x and y are natural numbers greater than 1, and $xy = 91$. Since 91 factors only as 7×13, and $x \geq y$, we must have $x = 13$ and $y = 7$.

4.4 Problem 4

The factorisations of 100 are

$$1 \times 100,\ 2 \times 50,\ 4 \times 25,\ 5 \times 20, \text{ and } 10 \times 10.$$

The first corresponds to a path, not a patio, so there are only 4 possible shapes. The smallest possible perimeter is $40m$, corresponding to the square $10m \times 10m$, and the largest is $104m$, corresponding to the dimensions $50m \times 2m$.

4.5 Problem 5

We have $x + y = 15$, so since $x \geq y > 1$ the possible values of x are the 6 integers from 8 to 13. The corresponding areas $x(15 - x)$ are 56, 54, 50, 44, 36, and 26. The largest area is $56m^2$, corresponding to $7m \times 8m$, and the least is $26m^2$, corresponding to $13m \times 2m$.

4.6 Problem 6

We have $xy = 186$, and the possible factorisations $x \times y$ of 186 are 93×2, 62×3, and 31×6. Only the last one gives a perimeter $2(x + y)$ less than $120m$, so the dimensions must be $31m \times 6m$.

4.7 Problem 7

We know to start with that x and y are positive integers, with $x \geq y > 1$. Let $s = x + y$ and $A = xy$. John knows A and Mary knows s.

Analysing the dialogue, we draw conclusions in order:

Mary's first remark tells us (and John) that $x + y < 76$.

John's first response tells Mary that there is more than one way to factorise A as the product uv of two natural numbers $u \geq v > 1$ with $u+v < 76$. In particular, A is not the product of two primes.

It is convenient to introduce some special terminology: Let us say that $a = u \times v$ is an *admissible factorisation* of the number a if u and v are natural numbers such that $u \geq v > 1$ and $u+v < 76$. Using this terminology, we can say that John's first response tells Mary that his A has at least two distinct admissible factorisations.

Mary's reply is very revealing. It tells John that no matter how you write $s = u + v$, with natural numbers $u \geq v > 1$, there will always be a second admissible factorisation for uv (apart from $u \times v$).

For a start, this means that s cannot be written as the sum of two primes. Now every even number less than 76 is the sum of two primes[1] as is easily, but rather tediously, checked by referring to the list of primes. So at this stage, John knows that s is odd.

What else does he know? He knows that s is at least 4, and is not $p + 2$, for any prime, so this rules out 5,7,9,13,15,19,21,25, 31,33,39, and some other numbers. But he also knows that s cannot take the form $37+e$, with an even number $e = 2z$ between 4 and 38, because $37 \times 2z$ has only one admissible factorisation. So s cannot be an odd number bigger than 40.

So John knows that s belongs to the set

$$M = \{11, 17, 23, 27, 29, 35, 37\}.$$

This is all the information that it is possible for John to extract from Mary's reply, because each of these numbers can be written in the form $u + v$, with $u \geq v > 1$, where uv admits a second admissible factorisation. For instance, $11 = 9 + 2$ and $9 \times 2 = 6 \times 3$. You can check the others.

John's next statement tells Mary that the value of $m = xy$ allows him to determine x and y, once he knows that $s = x + y$ belongs to M.

[1] The famous Goldbach Conjecture, dating from the eighteenth century, asserts that every even number whatsoever, no matter how large, is the sum of two primes. To prove this is a famous unsolved problem.

Now Mary can make a list of the possible pairs (x, y) corresponding to each $s \in M$, and calculate the list of products xy that might correspond to a given s:

s=11:

(x,y)	(9,2)	(8,3)	(7,4)	(6,5)
xy	18	24	28	30

s=17:

(x,y)	(15,2)	(14,3)	(13,4)	(12,5)	(11,6)	(10,7)	(9,8)
xy	30	42	52	60	66	70	72

and so on.

John's assertion, that he now knows the dimensions, tells Mary that his product xy occurs in only one of these lists. So she can rule out any product that occurs in two lists. For instance, she can rule out $xy = 30$. But her final assertion that she now knows the dimensions tells *us* that, after ruling out all such products, there is only one product left in the list corresponding to her s!

Denoting by L_s the list of products for $x + y = s$, we calculate:

$$\begin{aligned}
L_{11} &= \{18, 24, 28, 30\} \\
L_{17} &= \{30, 42, 52, 60, 66, 70, 72\} \\
L_{23} &= \{42, 60, 76, 90, 102, 112, 120, 126, 130, 132\} \\
L_{27} &= \{50, 72, 92, 100, 120, 140, 152, 162, 170, 176, 180, 182\} \\
L_{29} &= \{54, 78, 100, 120, 138, 154, 168, 180, 190, 198, 204, 208, 210\} \\
L_{35} &= \{66, 96, 124, 150, 174, 196, 216, 234, 250, 264, \\
&\qquad 276, 286, 294, 300, 304, 306\} \\
L_{37} &= \{70, 102, 132, 160, 186, 210, 232, 252, 270, 286, \\
&\qquad 300, 312, 322, 330, 336, 340, 342\}
\end{aligned}$$

Ruling out those that occur in multiple lists leaves more than one in all but L_{17}, which is left with just the element 52, so we conclude that $s = 17$ and $A = 52$, so that the dimensions are $13m \times 4m$.

One could make this problem more difficult, without changing the final answer, by raising the limit on the perimeter from 150 to 200, or even 400. In fact, we conjecture that the solution will not change if the upper limit is removed completely. However, the problem becomes ill-posed if we reduce the limit. This is illustrated by the next problem.

4.8 Problem 8

This time, Mary's first remark tells John that $s < 60$. Initially, the analysis proceeds as before, with 76 replaced by 60. The definition of 'admissible factorization' has to be changed, by replacing 76 by 60.

After Mary's second statement, John knows that s is at least 4, is not even, and is not $p + 2$ for any prime. But, this time, he also knows that s is not $31 + e$ for any even $e \geq 4$, because, were $s = 31 + e$, it would have only one admissible factorization. So John knows that s belongs to the set

$$M_1 = \{11, 17, 23, 27, 29\},$$

and this is all the information that he can extract from Mary's statements up to that point. The corresponding lists of products L_{11}, L_{17}, L_{23}, L_{27}, and L_{29} are the same as before.

John's next reply tells Mary that his product occurs in two of these lists.

But only the list L_{11} has just one product that appears in other lists as well, namely 30, so Mary's next statement tells *us* that $s = 11$ and $A = 30$, so we deduce that the dimensions are $6m \times 5m$.

4.9 Problem 9

The analysis proceeds precisely as in the previous problem, up to the point where John's second statement tells Mary that his product occurs on two of the 5 lists L_{11}, L_{17}, L_{23}, L_{27}, and L_{29}.

Mary's response tells John that there is more than one product on the list L_s corresponding to Mary's sum $s = x + y$ that also occurs on other lists. To be clear about this, let us introduce another piece of terminology. Let us say that a product is *ambiguous* if it occurs on at least two of the 5 lists, and is *unambiguous* otherwise.

What we (and John and Mary) know at this stage is that A is ambiguous, and that L_s has more than one ambiguous product in it.

John now says that he knows the dimensions, so that tells us that his A occurs on only one list that has more than one ambiguous product on it.

Now all the ambiguous products except 30 occur on two lists that have more than one ambiguous product, so we deduce that $A = 30$ and $s = 17$, so that the dimensions are $15m \times 2m$.