INTEGRAL BINOMIAL COEFFICIENTS

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Abstract. We give a short proof, using topology, of a fact about the denominators of certain binomial coefficients.

1. Introduction

The binomial coefficients are defined by

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!},$$

for nonnegative integral k and any α. Usually, α is a real or complex number, but the definition makes sense if α belongs to any field of characteristic zero. The following is well-known:

Theorem 1. *The binomial coefficients* $\binom{n}{k}$ *are positive integers, for integers* n, k *with* $0 \le k \le n$. □

The usual proof uses the Law of Pascal's Triangle, and induction.

The binomial coefficients $\binom{r}{k}$, with rational r, occur in the Maclaurin series expansion of $(1+x)^r$ (convergent for real or complex x with $|x| < 1$). For instance,

$$\sqrt{1+x} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k.$$

Calculating a few terms, one finds that the series begins

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{64}x^4 \cdots .$$

The coefficients are not integral (or nonnegative), but when common factors are cancelled (i.e. they are expressed in *reduced form* m/n, with $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and $\gcd(m,n) = 1$), it is remarkable that only powers of 2 occur in the denominators. This is not an accident: the pattern continues forever. We have the following, slightly less well-known result:

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Theorem 2. *Let $r \in \mathbb{Q}$ and $0 \le k \in \mathbb{Z}$. Suppose that $r = m/n$ in reduced form. Then the binomial coefficient $\binom{r}{k}$ has reduced form s/t, where t is a product of powers of primes that divide n.*

For instance, in the expansion of $(1+x)^{\frac{5}{6}}$, the coefficients all take the form $s/(2^a 3^b)$, for some $s \in \mathbb{Z}$.

The theorem may be proved using elementary number theory, for instance by reducing it to the statement that if $d, k \in \mathbb{N}$ and r is the largest factor of $k!$ prime to d, then r divides the product of the terms of each k-term arithmetic progression of integers having step d.

The purpose of this paper is to give a very short soft proof of Theorem 2, by using a little analysis (cf. Section 2).

2. A p-adic Proof

For prime $p \in \mathbb{N}$, let $\mathbb{Z}_p$ denote the ring of p-adic integers [1], and $\mathbb{Q}_p$ the field of p-adic numbers, the quotient field of $\mathbb{Z}_p$. The field $\mathbb{Q}$ is a subfield of $\mathbb{Q}_p$, and $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic metric. Each integer $m \in \mathbb{Z}$ belongs to $\mathbb{Z}_p$, and $\mathbb{Z}_p$ is the closure of $\mathbb{N}$ in $\mathbb{Q}_p$. A rational number r with reduced form m/n belongs to $\mathbb{Z}_p$ if and only if p does not divide n.

Theorem 3. *If $p \in \mathbb{N}$ is prime, $a \in \mathbb{Z}_p$ and $0 < k \in \mathbb{Z}$, then $\binom{a}{k} \in \mathbb{Z}_p$.*

Proof. Fix $k \in \mathbb{Z}$, $k \ge 0$. The function

$$f : x \mapsto \binom{x}{k}$$

is a polynomial with coefficients in $\mathbb{Q}$, and hence it is continuous, as a function from $\mathbb{Q}_p$ into $\mathbb{Q}_p$. (This just depends on the fact that $\mathbb{Q}_p$ is a metric field.) Choose a sequence $(a_n) \subset \mathbb{N}$ with $a_n \to a$ in p-adic metric. Then $f(a_n) \in \mathbb{N} \subset \mathbb{Z}_p$, and hence $f(a) = \lim_n f(a_n) \in \mathbb{Z}_p$, since $\mathbb{Z}_p$ is closed. □

We remark that a rational number r is an integer if and only if $r \in \mathbb{Z}_p$ for each prime p, and so this theorem may be regarded as a 'local version' of Theorem 1. The proof shows that the local version follows at once from Theorem 1, and a simple bit of topology.

Proof of Theorem 2. Let $r = m/n$, k, and $\binom{r}{k} = s/t$ be as in the statement. Suppose a prime p divides t. If p does not divide n, then $r \in \mathbb{Z}_p$, so $s/t \in \mathbb{Z}_p$, which is false. Thus each prime that divides t divides n. □

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References

[1] J.-P. Serre, A Course of Arithmetic. Springer. New York. 1996.

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