SUPPLEMENT TO FORMS OF NICE QUESTIONS

ANTHONY G. O'FARRELL

Abstract. This document is a supplement to the paper *Forms of Nice Questions*. We provide detail on the Sage code used.

1. Sage code

My aim here is to provide enough detail on the Sage code used in my paper *Forms of Nice Questions* [2] to allow an interested reader to replicate the derivation of the formulas.

Sage code may be executed remotely by using any web browser and going to the Sagemath cell site [1]. Sage is an extension of python, and the code looks like python code.

1.1. **Question 1.** For example, the following code evaluates the solution $f(2024)$ to Question 1:

```
Code:

var('t')
def f(t):
    return (t^2-1)/(t^2+1)
f(2024)

Result:
4096575/4096577
```

1.2. **The group determinant for S_3.** Here is the Sage code and result for the determinant (corresponding to the action of the permutations

$$\text{identity}, (12), (13), (23), (123), (321)$$

acting on the right on the group algebra, where $\sigma\tau$ means τ after σ):

$$\begin{vmatrix} a & b & c & d & e & f \\ b & a & f & e & d & c \\ c & e & a & f & b & d \\ d & f & e & a & c & b \\ f & d & b & c & a & e \\ e & c & d & b & f & a \end{vmatrix}:$$

Date: April 4, 2025.

```
Code:

var ('a,b,c,d,e,f');

A4 = matrix(SR,6,6,
       [a,b,c,d,e,f,b,a,f,e,d,c,c,e,a,f,b,d,
       d,f,e,a,c,b,f,d,b,c,a,e,e,c,d,b,f,a]);

A4.determinant().expand().factor();

Result:
(a^2-b^2+b*c-c^2+b*d+c*d-d^2-a*e+e^2-a*f-e*f+f^2)^2
*(a + b + c + d + e + f)*(a - b - c - d + e + f)
```

For the first example involving S_3, if we write out the linear system obtained from the equation

$$ah(t) + bh(1-t) + ch\left(\frac{1}{t}\right) + dh\left(\frac{1}{1-t}\right) + eh\left(\frac{t}{t-1}\right) + fh\left(\frac{t-1}{t}\right) = F(t),$$

it has the matrix

$$\begin{pmatrix} a & b & c & d & e & f \\ b & a & d & c & f & e \\ c & f & a & e & d & b \\ f & c & e & a & b & d \\ e & d & f & b & a & c \\ d & e & b & f & c & a \end{pmatrix}.$$

To evaluate its determinant when $(a,b,c,d,e,f) = (1,2,3,5,4,6)$:

```
var ('a,b,c,d,e,f');

A4 = matrix(SR,6,6,
        [a,b,c,d,e,f,
         b,a,d,c,f,e,
         c,f,a,e,d,b,
         f,c,e,a,b,d,
         e,d,f,b,a,c,
         d,e,b,f,c,a]);

print(A4.determinant().expand().factor())

a=1;b=2;c=3;d=4;e=5;f=6
(a^2 - b^2 + b*c - c^2 - a*d + d^2 + b*e + c*e - e^2 - a*f - d*f + f^2)^2*(a + b + c + d +

Result:
(a^2 - b^2 + b*c - c^2 - a*d + d^2 + b*e + c*e - e^2 - a*f - d*f + f^2)^2
```

```
*(a + b + c + d + e + f)*(a - b - c + d - e + f)

3024
```

(Representing (12) by $1-t$ and (13) by $1/t$, and composing as required by the setup, gives the correspondences

$$(13) \to \frac{t}{t-1},\ (123) \to \frac{1}{1-t},\ (321) \to \frac{t-1}{t},$$

and this is why the coefficients d and e have been exchanged, compared to the form given on the previous page.)

For the determinant corresponding to the equation

$$(1+t)h(t) + (1-t)h(1-t) + \frac{1}{t}h\left(\frac{1}{t}\right) = F(t),$$

in which the entries are rational functions, the sage is:

```
Code:

var('t')
A5 = matrix(SR,6,6,[1+t,1-t,1/t,0,0,0,
        t,1+1-t,0,1/(1-t),0,0,
        t,0,1+1/t,0,0,1-1/t,
        0,1-t,0,1+1/(1-t),t/(t-1),0,
        0,0,0,1/(1-t),1+t/(t-1),1-1/t,
        0,0,1/t,0,t/(t-1),1+(t-1)/t]);

#print(A5); #uncomment to view the matrix.
A5.determinant().expand().factor()

Result:
-4

Alternative code, which exposes the structure:
var('t')
def a(t):
    return 1-t
def b(t):
    return 1/t

A = matrix(6,6,
    [t + 1, a(t),  b(t),0,0,0,
    t, 1+a(t), 0,0, b(a(t)), 0,
    t, 0,  1+b(t), a(b(t)),0,0,
    0, 0, b(t),  1+a(b(t)),0, b(a(b(t))),
    0, a(t), 0, 0,  1+b(a(t)),a(b(a(t))) ,
    0,0,0, a(b(t)),  b(a(t)), 1+b(a(b(t)))
```

```
    ]                        )
#print(A) #Note: the order of the group elements has been changed
factor(simplify(A.determinant()))

Result:
-4
```

1.3. **The quaternion 8-group.** The code for the group determinant is:

```
Code:

A4 = matrix(SR,8,8,[b1,b2,b3,b4,b5,b6,b7,b8,
        b2,b1,b4,b3,b6,b5,b8,b7,
        b4,b3,b1,b2,b7,b8,b6,b5,
        b3,b4,b2,b1,b8,b7,b5,b6,
        b6,b5,b8,b7,b1,b2,b3,b4,
        b5,b6,b7,b8,b2,b1,b4,b3,
        b8,b7,b6,b5,b3,b4,b1,b2,
        b7,b8,b5,b6,b4,b3,b2,b1]);


A4.determinant().expand().factor();

Result (with line breaks added):
(b1^2-2*b1*b2+b2^2+b3^2-2*b3*b4+b4^2+b5^2-2*b5*b6+
b6^2+b7^2-2*b7*b8+b8^2)
*(b1^2-2*b1*b2+b2^2-b3^2+2*b3*b4-b4^2-b5^2+2*b5*b6-
b6^2+b7^2-2*b7*b8+b8^2)
*(b1 + b2 + b3 + b4 + b5 + b6 + b7 + b8)
*(b1 + b2 + b3 + b4 - b5 - b6 - b7 - b8)
*(b1 + b2 - b3 - b4 + b5 + b6 - b7 - b8)
*(b1 + b2 - b3 - b4 - b5 - b6 + b7 + b8)
```

1.4. **Question 4.** The nice round number comes about mainly because this group determinant, evaluated at $1, 2, 4, 8$, equals the current date, Anno Domini, 2025.

```
Code:
A = matrix(4,4,[1,2,4,8,
                2,1,8,4,
                4,8,1,2,
                8,4,2,1]
           )
A.determinant()
```

```
Result:
2025

#So Cramer's Rule tells us that h(t) is obtained by replacing
#the first column by the t^2 composed with the group elements.

Code:
var('t')
A = matrix(4,4,[t^2,2,4,8,
                t^2,1,8,4,
                1/t^2,8,1,2,
                1/t^2,4,2,1]
           )
factor(A.determinant())`

Result:
-45*(t^2 + 2)*(t^2 - 2)/t^2
```

Substituting $t = 3$, this becomes $-5 \times 77 = -385$.

References

[1] J Grout, I Hanson, S Johnson, A Kramer, A Novoseltsev, and W Stein. Sagemathcell. https://sagecell.sagemath.org/, 2024. Accessed: 2024 Oct 8.

[2] A G O'Farrell. Forms of nice questions. *College Math J*, to appear.

Mathematics and Statistics, Maynooth University, Co Kildare, Ireland W23 HW31
Email address: anthony.ofarrell@mu.ie