Greenberg's Betweenness Axioms

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1 Introduction

Greenberg [**?**] proposed axioms for Euclidean geometry which were a variation on Hilbert's [**?**]. Subsets of his axioms describe various non-categorical theories. This note is about the theory determined by the axioms of incidence and betweenness, which we now give.

The undefined terms are *point, line, incident with*, and *between*. We abbreviate 'B is between A and C' to $A * B * C$, and we employ the usual synonyms for 'incident with'. For instance 'A lies on l' and 'l passes through A' mean l is incident with A, and we employ terms such as collinear, with the usual definitions in terms of incident with.

The incidence axioms are:

I–1: For every point P and every fixed point Q not equal to P there exists a unique line l passing through P and Q.

I–2: For every line l there exist at least two distinct points that lie on l.

I–3: There exist three distinct points with the property that no line passes through all three of them.

If A and B are distinct points, we define $\overleftrightarrow{AB}$ to be the line passing through A and B, and $\{\overleftrightarrow{AB}\}$ to be the set of all points that lie on it.

The first three betweenness axioms are:

B–1: If $A * B * C$, then A, B and C are three distinct points all lying on the same line, and $C * B * A$.

B–2: Given any two distinct points B and D, there exists points A, C and E lying on $\overleftrightarrow{BD}$ such that $A * B * D, B * C * D$, and $B * D * E$.

B–3: If A, B and C are three distinct collinear points, then one and only one of them is between the other two.

We define the *segment* AB between two distinct points A, B to be the set of all points P such that $P = A, P = B$, or $A * P * B$.

Given a line l and points A, B not lying on l, we say that A *is on the same side of* l *as* B (ABl, for short) if $A = B$ or no point of AB lies on l. If A is not on the same side of l as B, we say that A *is on an opposite side of* l *to* B (AlB, for short).

The last betweenness axiom is:

B–4: For every line l, and any three points A, B, C not lying on l:

(i) If ABl and BCl, then ACl.

(ii) if AlB and BlC, then ACl.

We call the theory, determined by these undefined terms, definitions and axioms, *Greenberg's Betweenness Geometry.*

The purpose of this note is to show that the betweenness axioms given are independent. It is well-known that there are models of the geometry, so it suffices to give, for each $n \in \{1, 2, 3, 4\}$ a model of incidence geometry (i.e. satisfying I–1, I–2, I–3) in which B–n fails but the other betweenness axioms hold.

2 Independence of B–1

Consider the three-point model, with just 3 points P, Q, R and just 3 lines a, b, c such that Q and R lie on a, R and P lie on b, and P and Q lie on c. Interpret $A * B * C$ as true for all points A, B, C (distinct or not). Then the incidence axioms hold. Axiom B–1 fails, because $P * P * P$. Axiom B–2 holds trivially, and axioms B–3 and B–4 hold vacuously.

3 Independence of B–2

Consider the three-point model, as above, but interpret $A*B*C$ as false for all points A, B, C. The incidence axioms hold, and B–1, B–3, and B–4 hold vacuously, whereas B–2 fails.

4 Independence of B–3

This is the interesting one, as models are harder to come by.

Let F be the field with 3 elements, and interpret a *point* as an ordered pair (x, y), with $x \in F$ and $y \in F$. Interpret a *line* as a set $\{(x, y) \in F^2 : ax + by = c\}$ where $a, b, c \in F$, and $(a, b) \neq (0, 0)$. Interpret point P incident with line l as $P \in l$. Interpret $A * B * C$ to mean that the points A, B, C are distinct and collinear.

In other words, the model is the 9-point affine plane with an undemanding betweenness relation on each line. Each line has precisely 3 points and there are two other lines parallel to (i.e. not intersecting) any given line.

The incidence axioms hold, and B–1. Axiom B–2 holds, because the third point on the line $\overleftrightarrow{BD}$ does for A, C and E. Axiom B–3 fails, obviously.

Given a line l and distinct points A, B not on it, one observes that ABl if and only if $\overleftrightarrow{AB}$ is parallel to l. From this it is easy to check that B–4 holds.

5 Independence of B–4

Take the usual $\mathbb{R}^3$ model of three-dimensional Euclidean geometry, with $A * B * C$ interpreted as usual for collinear A, B, C. Then the incidence axioms and B–1, B–2, B–3 hold, but one readily checks that B–4 fails.

Problems:

Is there a model in which all lines are infinite and only B–3 fails?

Is there a model in which only B–3 fails and there are distinct collinear points A, B, C of which none is between the other two?

References

[1] M.J. Greenberg, Euclidean and Non-Euclidean Geometrics. 3^{rd} Ed. Freeman. New York. 1994.

[2] D.Hilbert, The Foundation of Geometry. 3^{rd} Ed. (translated by E.J. Townsend). Open Court. La Salle, Illnois. 1938.