

LOCALNESS OF $A(\Psi)$ ALGEBRAS

By

G.R. ALLAN

DPMMS, 16 Mill Lane, Cambridge CB21SB, England

G. KAKIKO

Department of Mathematics, University of Dar-es-Salaam, Dar-es-Salaam, Tanzania

A.G. O'FARRELL AND R.O. WATSON

Department of Mathematics, NUI Maynooth, Co. Kildare, Ireland

ABSTRACT

Let d and r be positive integers. Given $\Psi = (\psi_1, \dots, \psi_r) \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$, we consider the unital algebra $\mathbf{R}[\Psi] = \mathbf{R}[\psi_1, \dots, \psi_r]$ generated by $\{\psi_1, \dots, \psi_r\}$, and its closure $A(\Psi)$ in C^∞ topology.

We identify the space of closed maximal ideals of $A(\Psi)$, we establish that it is a regular algebra, and we show that the approximation problem, to provide an explicit description of Ψ , is local to the level sets of Ψ .

1. Introduction.

Let d and r be positive integers, throughout. Given $\Psi = (\psi_1, \dots, \psi_r) \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$, we consider the unital algebra $\mathbf{R}[\Psi] = \mathbf{R}[\psi_1, \dots, \psi_r]$ generated by $\{\psi_1, \dots, \psi_r\}$, and its closure $A(\Psi)$ in C^∞ topology.

We recall that $A(\Psi)$ is also the closure of the algebra

$$C^\infty(\Psi) = \{g \circ \Psi : g \in C^\infty(\mathbf{R}^r, \mathbf{R})\}$$

(cf. [1,2] for background).

This paper is about the properties of $A(\Psi)$ *qua* Fréchet algebra, and the approximation problem: describe $A(\Psi)$ explicitly.

We identify the space of closed maximal ideals of $A(\Psi)$, we establish that it is a regular algebra, and we show that the approximation problem is local to the level sets of Ψ .

2. Associated Topologies.

We shall only use the Euclidean topology on \mathbf{R}^r , but we need to consider some other, *a priori* distinct topologies on \mathbf{R}^d .

Definition. We define the Ψ -*hull-kernel topology* on \mathbf{R}^d as that corresponding to the Kuratowski closure operation

$$E \mapsto \text{HK}(E) = \{b \in \mathbf{R}^d : f(b) = 0 \text{ whenever } f \in A(\Psi) \text{ and } f|_E = 0\}.$$

Remark. This is in general finer than the pull-back topology

$$\{\Psi^{-1}(U) : U \text{ open in } \mathbf{R}^r\}.$$

It may happen that $\text{clos}\Psi(E) \cap \text{clos}\Psi(F) \neq \emptyset$ for disjoint Ψ -hull-kernel closed sets E, F .

We abbreviate Ψ -hull-kernel topology to HK-topology when convenient. We denote the HK-closure of a set E by $\text{HK}(E)$. This is consistent with the following (more-or-less

standard) notation:

$$H(F) = \{a \in \mathbf{R}^d : f(a) = 0, \forall f \in F\}, \forall F \subset A(\Psi);$$

$$K(E) = \{f \in A(\Psi) : f(a) = 0, \forall a \in E\}, \forall E \subset \mathbf{R}^d.$$

Note that

$$HKH(F) = H(F), \forall F \subset A(\Psi),$$

$$KHK(E) = K(E), \forall E \subset \mathbf{R}^d.$$

In particular, $H(F)$ is HK-closed, for each $F \subset A(\Psi)$.

We note that each HK-closed set is a union of level sets of Ψ , that $\text{HK}(\{a\}) = \Psi^{-1}(\Psi(a))$ (the level set through the point a), and that the minimal nonempty HK-closed sets are these level sets of Ψ . We abbreviate $\text{HK}(\{a\})$ to $\text{HK}(a)$.

Definition. We define the Ψ -weak-star topology on \mathbf{R}^d as the pull-back topology corresponding to the weak-star topology on the dual $A(\Psi)^*$ and the natural injection of \mathbf{R}^d into $A(\Psi)^*$.

In other words, the set N is a weak-star neighbourhood of the point $a \in \mathbf{R}^d$ if and only if there exist a finite number of functions f_1, \dots, f_n belonging to $A(\Psi)$, such that

$$\{x \in \mathbf{R}^d : |f_j(x) - f_j(a)| < 1, \forall j\} \subset N.$$

Since we are dealing here with real-valued functions, it is evident that it makes no difference if we insist that n always equal 1. In fact, it is easy to see that the set N is a weak-star neighbourhood of the point $a \in \mathbf{R}^d$ if and only if there exists a function $f \in A(\Psi)$, such that $f(a) = 0$ and

$$\{x \in \mathbf{R}^d : f(x) < 1\} \subset N.$$

We abbreviate Ψ -weak-star topology to WS-topology, when convenient.

Unqualified topological terms (open, closed, ...) refer to the Euclidean topology. It is clear that the WS-topology is at least as fine as the HK-topology, and the Euclidean topology is at least as fine as the WS-topology. We shall discover more below.

We denote the closed ball with centre $x \in \mathbf{R}^d$ and radius $r \geq 0$ by $\mathbf{B}(x, r)$, and the corresponding open ball by $\mathbf{U}(x, r)$.

Proposition 1. *Let $\Psi \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$. Suppose E and F are disjoint closed subsets of \mathbf{R}^d , $\Psi^{-1}\Psi(E) = E$, and $\Psi^{-1}\Psi(F) = F$. Then there exists $f \in A(\Psi)$ such that $f = 1$ on E and $f = 0$ on F .*

PROOF. Let E and F be as in the hypothesis. Define

$$B_n = \mathbf{B}(0, n), \quad B'_n = \Psi(B_n),$$

$$E_n = E \cap B_n, \quad E'_n = \Psi(E_n),$$

$$F_n = F \cap B_n, \quad F'_n = \Psi(F_n),$$

$$H_n = B_{n-1} \cup E_n \cup F_n, \quad H'_n = \Psi(H_n),$$

We observe that E'_n, F'_n , and H'_n are compact subsets of \mathbf{R}^r , and

$$E'_n \cap F'_n = \emptyset,$$

$$E_{n+1} \cap B_n = E_n,$$

$$E'_{n+1} \cap B'_n = E'_n, \quad (\text{because } E = \Psi^{-1}\Psi(E))$$

$$F'_{n+1} \cap B'_n = F'_n.$$

Choose $g_1 \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that $g_1 = 1$ near E'_1 and $g_1 = 0$ near F'_1 .

Consider the function $g_2 : H'_2 \rightarrow \mathbf{R}$ given by

$$g_2(x) = \begin{cases} g_1(x), & x \in B'_1, \\ 1, & x \in E'_2, \\ 0, & x \in F'_2, \end{cases}$$

This is well-defined, since $g_1 = 1$ on $B'_1 \cap E'_2$ and $g_1 = 0$ on $B'_1 \cap F'_2$. Moreover, each point of H'_2 has a neighbourhood to which g_2 has a C^∞ extension; indeed one of g_1 , 1 or 0 will do as the extension. Since the existence of a global C^∞ extension is a local property, it follows that g_2 has an extension in $C^\infty(\mathbf{R}^r, \mathbf{R})$, and we denote such an extension by the same symbol, g_2 . We may choose g_2 so that it is 1 near E'_2 and 0 near F'_2 .

Continuing in this way, we find $g_{n+1} \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that

$$g_{n+1}|_{B'_n} = g_n|_{B'_n},$$

$$g_{n+1} = 1 \text{ near } E'_{n+1} \text{ and } g_{n+1} = 0 \text{ near } F'_{n+1}.$$

Define $f : \mathbf{R}^d \rightarrow \mathbf{R}$ by setting

$$f|_{B_n} = g_n \circ \Psi|_{B_n}, \forall n.$$

Evidently, f is well-defined, $f \in C^\infty(\mathbf{R}^d, \mathbf{R})$, and $g_n \circ \Psi \rightarrow f$ in C^∞ topology as $n \uparrow +\infty$.

Thus $f \in A(\Psi)$. Finally, it is clear that $f = 1$ on E and $f = 0$ on F , so we are done. ■

Corollary 2. *Suppose $E \subset \mathbf{R}^d$. Then E is HK-closed if and only if E is closed and $\Psi^{-1}\Psi(E) = E$.*

PROOF. The ‘only if’ part is obvious. For the converse, suppose that E is closed and $\Psi^{-1}\Psi(E) = E$. Let $a \notin E$. Then $F = \Psi^{-1}\Psi(a)$ is closed and disjoint from E , and $\Psi^{-1}\Psi(F) = F$. By the Proposition, there exists $f \in A(\Psi)$ such that $f|_E = 0$ and $f|_F = 1$. Thus $a \notin \text{HK}(E)$. This shows that $\text{HK}(E) \subset E$. Evidently $E \subset \text{HK}(E)$, so $\text{HK}(E) = E$, and we are done. ■

Corollary 3. *Let $E \subset \mathbf{R}^d$. Then $HK(E)$ is the least set $F \subset \mathbf{R}^d$ such that $E \subset F$, F is closed, and $\Psi^{-1}\Psi(F) \subset F$. ■*

This fact implies that the HK-closure of a set E may be obtained by forming $E_0 = E$, and proceeding by transfinite induction:

$$E_{\alpha+1} = \text{clos}\Psi^{-1}\Psi(E_\alpha), \forall \text{ ordinals } \alpha,$$

$$E_\alpha = \bigcup_{\beta < \alpha} E_\beta, \forall \text{ limit ordinals } \alpha,$$

until the first ordinal having cardinal greater than the continuum, at the worst.

Corollary 4. *Let $\Psi \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$. Then the following three conditions are equivalent:*

- (1) Ψ is injective.
- (2) The HK-topology is Hausdorff.
- (3) The HK-topology is the same as the Euclidean topology.

PROOF. Obviously (3) \Rightarrow (2) \Rightarrow (1). The only delicate point is (1) \Rightarrow (3), and this is immediate from Corollary 2. ■

Finally, we note that combining Proposition 1 and Corollary 2, we have:

Corollary 5. *Let E and F be disjoint HK-closed sets. Then there exists $f \in A(\Psi)$ such that $f = 1$ on E and $f = 0$ on F . ■*

Now we consider the C^∞ action and its consequences.

Proposition 6. *Let $h \in C^\infty(\mathbf{R}^r, \mathbf{R}^p)$. Then the (usually nonlinear) map*

$$h \circ : \begin{cases} C^\infty(\mathbf{R}^d, \mathbf{R}^r) \rightarrow C^\infty(\mathbf{R}^d, \mathbf{R}^p), \\ f \mapsto h \circ f, \end{cases}$$

is continuous.

PROOF. This is immediate from Faa di Bruno's formula [4, p.222]. ■

Corollary 7. *$C^\infty(\mathbf{R}, \mathbf{R})$ acts by composition on $A(\Psi)$, i.e. $f \mapsto h \circ f$ maps $A(\Psi)$ into itself, whenever $h \in C^\infty(\mathbf{R}, \mathbf{R})$.*

PROOF. Let $f \in A(\Psi)$. Choose $g_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that $g_n \circ \Psi \rightarrow f$ in $C^\infty(\mathbf{R}^d, \mathbf{R})$ topology. By the Proposition,

$$(h \circ g_n) \circ \Psi = h \circ (g_n \circ \Psi) \rightarrow h \circ f$$

in C^∞ topology. Thus $h \circ f \in A(\Psi)$. ■

Corollary 8. *Let $h \in C^\infty(\mathbf{R}, \mathbf{R})$, with $h(0) = 0$. Then $h \circ$ maps each ideal $I \subset A(\Psi)$ into itself.*

PROOF. We may factorize $h(x)$ as $xk(x)$, where $k \in C^\infty(\mathbf{R}, \mathbf{R})[\mathbf{T}]$.

Let $f \in I$. Then, using Corollary 2, we get

$$h \circ f = f \cdot (k \circ f) \in f \cdot A(\Psi) \subset I,$$

as required. ■

Corollary 9. *Let $g \in A(\Psi)$. Then the set*

$$U = \{a \in \mathbf{R}^d : g(a) > 0\}$$

is HK–open.

PROOF. Choose $h \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $h(x) > 0$ whenever $x > 0$ and $h(x) = 0$ whenever $x \leq 0$.

By Corollary 7, $h \circ g \in A(\Psi)$, and evidently

$$\mathbf{R}^d \sim U = H(\{h \circ g\})$$

is HK–closed. This suffices. ■

Corollary 10. *The HK–topology is the same as the WS–topology.*

PROOF. It follows readily from Corollary 9 that the set

$$\{x \in \mathbf{R}^d : f(x) < 1\}$$

is HK–open, whenever $f \in A(\Psi)$. Since the sets of this form, corresponding to $f \in A(\Psi)$ with $f(a) = 0$, form a neighbourhood base for the point $a \in \mathbf{R}^d$, we conclude that each WS–open set is HK–open, and this suffices. ■

We summarize our characterizations of the HK–topology, adding a useful converse to Corollary 9:

Theorem 11. *Let $U \subset \mathbf{R}^d$. Then the following are equivalent:*

- (1) U is HK–open.
- (2) U is WS–open.
- (3) U is open and $\Psi^{-1}\Psi(U) = U$.
- (4) There exists $g \in A(\Psi)$ such that

$$U = \{x \in \mathbf{R}^d : g(x) > 0\}.$$

PROOF. In view of Corollaries 2, 9 and 10, it only remains to prove that (1) implies (4).

Let $F = \mathbf{R}^d \sim U$.

Given $n \in \mathbf{N}$, consider $B_n = \mathbf{B}(0, n)$ and

$$E_n = \left\{ x \in B_n : \text{dist}(x, F) \geq \frac{1}{n} \right\}.$$

For each $a \in E_n$, there exists $f_a \in A(\Psi)$ such that $f_a(a) = 1$ and $f_a = 0$ on F . Thus the set

$$N_a = \{x \in \mathbf{R}^d : f_a(x) > 0\}$$

is a HK–neighbourhood of a . Since E_n is compact, we may choose $a_1, a_2, \dots, a_m \in E_n$ such that N_{a_1}, \dots, N_{a_m} cover E_n . Let

$$g_n = f_{a_1}^2 + \dots + f_{a_m}^2.$$

Then $g_n \in A(\Psi)$, $g_n = 0$ on F , $g_n \geq 0$ on \mathbf{R}^d , and $g_n > 0$ on E_n . Let

$$M_n = 1 + \max_{|i| \leq n} \sup_{B_n} |\partial^i g_n|.$$

Define

$$g = \sum_{n=1}^{\infty} \frac{g_n}{2^n M_n}.$$

For any given $m \in \mathbf{N}$ and $k \in \mathbf{Z}_+$, we have

$$\sup_{B_m} \left| \partial^i \left(\frac{g_n}{2^n M_n} \right) \right| \leq 2^{-n}, \forall |i| \leq k \quad \forall n \geq m.$$

Thus the series for g converges in C^∞ topology, and $g \in A(\Psi)$.

Evidently, $g > 0$ on U and $g = 0$ on F , so we are done. \blacksquare

3. Maximal Ideals.

Proposition 12. *Suppose $f \in A(\Psi)$ and $f(a) \neq 0, \forall a \in \mathbf{R}^d$. Then $1/f \in A(\Psi)$.*

PROOF. Fix K compact in \mathbf{R}^d and $k \in \mathbf{Z}_+$. It suffices to show that there exist $g_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that

$$\partial^i(g_n \circ \Psi) \rightarrow \partial^i \left(\frac{1}{f} \right), \forall |i| \leq k,$$

uniformly on K .

We may assume that K is a ball, without loss in generality.

Choose $h_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that

$$\partial^i(h_n \circ \Psi) \rightarrow \partial^i(f), \forall |i| \leq k,$$

uniformly on K . Since $f \neq 0$ on K , we may assume, without loss in generality, that $f > 0$ on K (since K is connected). Let $\kappa = \inf_K f$.

Discarding some initial terms of the sequence (if need be), we may assume that $h_n > \kappa/2$ on $\Psi(K)$, for each n .

Choose $k_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that $k_n = \log h_n$ near K , and let $r_n = \exp(-k_n)$. Then $g_n = 1/h_n$ on $\Psi(K)$, and

$$\partial^i(g_n \circ \Psi) = \partial^i \left(\frac{1}{h_n \circ \Psi} \right) \rightarrow \partial^i \left(\frac{1}{f} \right), \forall |i| \leq k,$$

uniformly on K , as required. ■

Theorem 13. *Let $M \subset A(\Psi)$. Then the following are equivalent:*

- (1) M is a closed maximal ideal in $A(\Psi)$.
- (2) There exists $a \in \mathbf{R}^d$ such that $M = K(a)$.

PROOF. It is easy to see that (2) implies (1).

To prove that (1) implies (2), fix a closed maximal ideal M . We wish to show that M is the HK-closure of some point. Since (2) implies (1), it suffices to show that $H(M)$ is nonempty.

Suppose that $H(M) = \emptyset$.

For each $a \in \mathbf{R}^d$, we may choose $f_a \in M$ such that $f_a(a) = 1$. Using compactness, as in the proof of Proposition 12, we may choose for each $n \in \mathbf{N}$ a function $g_n \in M$ such that $g_n \geq 0$ on \mathbf{R}^d and $g_n > 0$ on B_n . Adding these up with suitable weights, we get $g \in \text{clos}(M) = M$ such that $g > 0$ on \mathbf{R}^d . By Proposition 12, $1/g \in A(\Psi)$, so $1 \in M$, so $M = A(\Psi)$, contradicting the maximality of M . This contradiction shows that $H(M) \neq \emptyset$, and we are done. ■

Remark. The argument of this proof actually shows that each ideal having empty hull is dense in $A(\Psi)$. From this observation, it is not hard to deduce that the maximal closed ideals are the same as the closed maximal ideals.

Example. Let $\Psi(x) = x, \forall x \in \mathbf{R}$, so that $A(\Psi) = C^\infty(\mathbf{R}, \mathbf{R})$. The subset of all functions having compact support is an proper ideal, and hence is contained in a maximal ideal M .

Since $H(M) = \emptyset$, M cannot be closed.

In fact, this $A(\Psi)$ has many dense maximal ideals, corresponding to some kind of ultrafilters.

Recall that a *character* of a real Fréchet algebra is, by definition, a nonzero algebra homomorphism from the algebra to \mathbf{R} . There is a bijective correspondence between characters and maximal ideals with quotient isomorphic to \mathbf{R} . It is known that all characters on real Fréchet algebras are necessarily continuous [3].

Corollary 14. *The characters of $A(\Psi)$ are the evaluations at the points of \mathbf{R}^d .*

PROOF. Since characters are continuous, the kernel of a character is a closed maximal ideal, hence is $K(a)$ for some $a \in \mathbf{R}^d$. It follows easily that the character is evaluation at a . ■

Characters belong to $A(\Psi)^*$, so the space of characters inherits the weak–star topology.

We may thus rephrase Corollary 5, as follows:

Theorem 15. *Let E and F be disjoint weak–star closed sets of characters on $A(\Psi)$. Then there exists a function $f \in A(\Psi)$ such that $\phi(f) = 1$ for all $\phi \in E$ and $\phi(f) = 0$ for all $\phi \in F$. ■*

This is the *regularity* referred to in the introduction.

4. Localness.

Segal asked in 1949 whether $A(\Psi)$ has a local description analogous to the Stone–Weierstrass Theorem. Nachbin conjectured that membership of f in $A(\Psi)$ is determined by the behaviour of f on each level set of Ψ . This conjecture may be reformulated in terms of Taylor series, and some special cases have been proved by Tougeron and the authors. For a more detailed account of the history, see [1, 2]. To date, it has not even been established that membership in $A(\Psi)$ depends only on the behaviour of f *near* each level set of Ψ . This we shall now do.

First, we establish a preliminary fact.

Lemma 16. *Let $E \subset U \subset \mathbf{R}^d$, where E is HK-closed and U is (Euclidean) open. Let K be compact. Then there exists a HK-open set V such that $E \subset V$ and $K \cap V \subset K \cap U$.*

PROOF. By Theorem 11, we may choose $h \in A(\Psi)$ such that $h = 0$ on E and $h > 0$ off E . Let $\eta = \inf_{K \sim U} h$. Then $\eta > 0$. Take $V = \{x \in \mathbf{R}^d : h(x) < \eta\}$. Then V is HK-open, by Corollary 9, $E \subset V$, and $K \cap V \subset K \cap U$, as required. ■

Theorem 17. *Let $\Psi \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$ and $f \in C^\infty(\mathbf{R}^d, \mathbf{R})$. Then the following four conditions are equivalent:*

- (1) $f \in A(\Psi)$;
- (2) $\forall a \in \mathbf{R}^d$, there exists a HK-neighbourhood U of a and there exists a function $g \in A(\Psi)$ such that $g = f$ on U .
- (3) $\forall a \in \mathbf{R}^d$, there exists a HK-open neighbourhood U of a and there exists a sequence of $g_n \in \mathbf{R}[\Psi]$ such that $g_n \rightarrow f$ in $C^\infty(U, \mathbf{R})$ topology.

(4) $\forall a \in \mathbf{R}^d$, and for each compact $K \subset \mathbf{R}^d$, there exists a (Euclidean) open neighbourhood U of $K \cap \text{HK}(a)$ and there exists a sequence of $g_n \in \mathbf{R}[\Psi]$ such that $g_n \rightarrow f$ in $C^\infty(U, \mathbf{R})$ topology.

PROOF. It is evident that (1) implies (2), (2) implies (3), and (3) implies (4).

(4) \Rightarrow (1): Suppose (4). Fix $K \subset \mathbf{R}^d$, compact, and $k \in \mathbf{Z}_+$.

For each $a \in K$, choose an open neighbourhood U_a of $K \cap \text{HK}(a)$ and a sequence $g_{a,n} \in \mathbf{R}[\Psi]$ such that $g_{a,n} \rightarrow f$ in $C^\infty(U_a, \mathbf{R})$ topology.

For each $a \in K$, Lemma 16 allows us to choose a HK–open set V_a such that $\text{HK}(a) \subset V_a$ and $K \cap V_a \subset K \cap U_a$.

By Corollary 5 we may choose $\rho_a \in A(\Psi)$ such that $\rho_a > 0$ precisely on V_a and $\rho = 1$ on $\text{HK}(a)$. Let $W_a = \{x \in \mathbf{R}^d : \rho_a(x) > 1/2\}$. Then W_a is HK–open and $\text{HK}(a) \subset W_a$.

By compactness, we may choose $a_1, \dots, a_m \in K$ such that $K \subset W_{a_1} \cup \dots \cup W_{a_m}$. Let us abbreviate W_{a_i} to W_i , V_{a_i} to V_i , and $g_{a_i,n}$ to $g_{i,n}$. Choose $h_i \in A(\Psi)$ such that $h_i > 0$ precisely on W_i .

Let $W = W_1 \cup \dots \cup W_m$. Since W is HK–open, we may choose $h \in A(\Psi)$ such that $h > 0$ precisely on W . Since K is compact, the number $\eta = \inf_K h$ is strictly positive. Let $F = \{h \geq \eta/2\}$. Then F and $\mathbf{R}^d \sim W$ are disjoint HK–closed sets, and $K \subset F$. Choose $h_0 \in A(\Psi)$ such that $h_0 > 0$ precisely on $\mathbf{R}^d \sim F$. Then $s = h_0 + \dots + h_m$ belongs to $A(\Psi)$ and $s > 0$ on \mathbf{R}^d , so $1/s \in A(\Psi)$, by Proposition 12. Let $k_i = h_i/s$. Then $k_i \in A(\Psi)$, $k_i \geq 0$, $\sum_0^m k_i = 1$, $k_0 = 0$ near K and $\text{spt} k_i \subset W_i$ whenever $i \geq 1$.

Fix i , $1 \leq i \leq m$. Let $T_i = K \cap \{\rho_i \geq 1/2\}$. Then T_i is a compact subset of $K \cap U_i$.

Since $k_i = 0$ on $K \sim T_i$, and $g_{i,n} \rightarrow f$ in $C^\infty(U_i, \mathbf{R})$, we see that

$$\partial^i(k_i \cdot g_{i,n} - k_i \cdot f) \rightarrow 0$$

uniformly on K , for each $|i| \leq k$, as $n \uparrow \infty$. Since $k_0 = 0$ near K , we conclude that the function $r_n = \sum_1^m k_i g_{i,n}$, which belongs to $A(\Psi)$, converges uniformly, along with all derivatives up to order k , uniformly on K , to $\sum_1^m k_i f$, which equals f on K . This suffices to show that (1) holds. ■

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