PERVASIVE ALGEBRAS OF ANALYTIC FUNCTIONS ON RIEMANN SURFACES

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ABSTRACT. Let U be an open subset on an open Riemann surface with clos U compact. We give necessary and sufficient conditions for U such that the algebra $A(U)$ is complex pervasive on bdy U. Complex pervasive means that the uniform closure on each proper closed subset E of bdy U is the space of all complex-valued continuous functions on E.

1. Introduction

Let X be a compact Hausdorff topological space and denote by $C(X)$ the Banach algebra of all complex-valued continuous functions with the uniform norm. A *function space* S on X is a closed subspace of $C(X)$. By $\text{clos}_{C(E)}\, S$, we denote the uniform closure on E of the function space S, where E is a closed subset of X.

Given a closed set $Y \subset X$, the function space S on X is said to be *complex pervasive* on Y if $\text{clos}_{C(E)}\, S = C(E)$ whenever E is a proper non-empty closed subset of Y.

Let U be an open subset of an open Riemann surface $\mathcal{R}$, and denote by bdy U its topological boundary. In this paper we shall consider the case where $X = \text{clos}\, U$, $Y = \text{bdy}\, U$ and S coincides with the algebra $A(U)$ of complex valued functions continuous on clos U and analytic on U.

The concept of pervasive spaces was introduced by Hoffman and Singer in 1960 [11] in relation with the study of maximal uniform algebras.

The real pervasiveness (analogously defined) of spaces of harmonic functions on Euclidean spaces was studied by Netuka in [13], where it is shown that if the open set $U \subset \mathbb{R}^d$ is bounded, connected and satisfies bdy U = bdy clos U, then the space of functions continuous on clos U and harmonic on U is real pervasive on bdy U.

The study of the complex and real pervasiveness of the algebras $A(U)$ where U is an open subset of the Riemann sphere $\hat{\mathbb{C}}$ has been treated by Netuka et al. in [14]. In this paper a complete characterization in topological terms of the complex pervasiveness of the algebra $A(U)$ on bdy U is given, as well as a complete characterization of the real pervasiveness of Re $A(U)$ (space of real parts of elements of $A(U)$). In the latter case, a topological characterization is not possible but one can be given involving continuous analytic capacity. This result draws on the characterization given by Gamelin and Garnett [8] of those U such that $A(U)$ is a Dirichlet algebra on bdy U [7].

The present note was prompted by the question, whether the results concerning complex pervasiveness in [14] could be extended to open Riemann surfaces. Using

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a *Cauchy transform* on the surface (see below) in the manner of Scheinberg [16] and Gauthier [9], many of the results concerning uniform holomorphic (respectively meromorphic) approximation can be carried from the plane to open Riemann surfaces. We show that this result on pervasiveness is no exception.

2. Preliminary results

The dual space $C(X)^*$ of $C(X)$, where X is a compact Hausdorff topological space will be identified with the space of complex regular Borel measures on X and denoted by $M(X)$. The (closed) support of a measure $\mu \in M(X)$ will be denoted by spt μ.

For a subset $S \subset C(X)$ and a measure $\mu \in M(X)$ we write $\mu \perp S$ and say μ *annihilates* S, if $\int f \, d\mu = 0$ whenever $f \in S$. The set of annihilating measures of S will be denoted by $S^\perp$.

As an easy consequence of the Hahn-Banach Theorem, and as remarked in [6], a subspace $S \subset C(X)$ is complex pervasive on Y, where Y is a closed subset of X, if and only if each $\mu \in S^\perp$, $\mu \neq 0$ has spt $\mu = Y$. Equivalently, S is complex pervasive on Y if and only if the conditions $\mu \in M(Y)$, $\mu \perp S$ and spt $\mu \subsetneq Y$ imply that $\mu = 0$.

Let $\mathcal{R}$ be a connected open Riemann surface. Gunning and Narashiman have shown that $\mathcal{R}$ can be visualized in a very concise way [10]. More precisely,

Theorem. *Any (connected) open Riemann surface $\mathcal{R}$ admits a holomorphic immersion into the complex plane; that is, there is a holomorphic mapping $\rho\colon \mathcal{R} \longrightarrow \mathbb{C}$ which is a local homeomorphism.*

Therefore ρ is a global uniformizing parameter on $\mathcal{R}$. A *parametric disc* $D(z,r)$ of center $z \in \mathcal{R}$ and radius $r > 0$ is an open set on $\mathcal{R}$ biholomorphic under ρ to the disc of center $\rho(z)$ and radius r on the complex plane.

Note that $d\rho$ is a globally nowhere zero holomorphic 1-form, so this global uniformizing parameter gives rise to an area element $dA = d\rho \wedge d\overline{\rho}$.

Given a compact set $K \subset \mathcal{R}$, $R(K)$ denote the algebra of functions in $C(K)$ which are uniform limits on K of meromorphic functions with poles off K, and $\mathcal{O}(K)$ the set of functions holomorphic in a neighbourhood of K. By the Runge-Behnke-Stein Theorem $R(K) = \operatorname{clos}_{C(K)} \mathcal{O}(K)$.

Using Gunning and Narashiman's result and the fact that $\mathcal{R}$ is Stein (so the first Cousin problem is solvable), Scheinberg [16] and Gauthier [9] constructed a globally defined meromorphic function $q(z,w)$, for $z, w \in \mathcal{R}$, such that $q(\cdot, w)$ has a simple pole at $z = w$ and locally $q(z,w) - (\rho(z) - \rho(w))^{-1}$ is a holomorphic function near $z = w$. For this reason $q(\cdot, w)$ is called a *Cauchy kernel* on $\mathcal{R}$. An application of Stokes Theorem gives the following Cauchy-Pompeiu Theorem [16].

Proposition 2.1. *Let U be an open set in $\mathcal{R}$ with* clos U *compact, having a piecewise C^1 oriented boundary and let $f \in C^1(\text{clos } U)$. Then for every $w \in U$,*

$$f(w) = \frac{1}{2\pi i} \int_{\text{bdy } U} f(z) q(z,w) \, d\rho(z) + \frac{1}{2\pi i} \int_U \frac{\partial f}{\partial \overline{\rho}}(z) q(z,w) \, dA(z).$$

In particular, if $f \in C^1(\text{clos } U) \cap \mathcal{O}(U)$, f satisfies the Cauchy formula

$$f(w) = \frac{1}{2\pi i} \int_{\text{bdy } U} f(z) q(z, w) \, d\rho(z).$$

Definition 2.2. Let μ be a complex measure with compact support on $\mathcal{R}$. The *q-Cauchy transform* of μ is defined by

$$\hat{\mu}(w) := \frac{1}{\pi} \int_{\mathcal{R}} q(z, w) \, d\mu(z).$$

Remark 2.3. The definition of the q-Cauchy transform depends on the meromorphic function q which is in general not unique. For the purpose of this paper we abreviate the q-Cauchy transform of μ to the Cauchy-transform of μ.

In local coordinates, $\hat{\mu}$ is the convolution of a locally integrable function and a measure with compact support, so $\hat{\mu}$ converges absolutely except for w in a set of A-measure zero.

Note also that as the Cauchy kernel q is analytic except at $z = w$, $\hat{\mu}$ is analytic outside spt μ. The analyticity of $\hat{\mu}$ follows by differentiation under the integral sign.

The following results are standard (cf. [7, p. 46], [5], [15]) but we include them for the convenience of the reader.

The importance of the Cauchy transform in approximation theory comes from the following lemma.

Lemma 2.4. *Let μ be a complex measure on a compact subset K of $\mathcal{R}$. The Cauchy transform $\hat{\mu}$ vanishes off K if and only if $\mu \perp R(K)$.*

Proof. If $\mu \perp R(K)$ then clearly, as $q(z, w)$ is analytic except at $z = w$, $\hat{\mu} = 0$ off K.

Conversely, let $f \in \mathcal{O}(K)$. Then by Proposition 2.1

$$f(w) = \frac{1}{2\pi i} \int_{\Gamma} f(z) q(z, w) \, d\rho(z),$$

where Γ is an appropriate contour around K.

Therefore

$$\begin{aligned} \int_{\mathcal{R}} f(w) \, d\mu(w) &= \int_{\mathcal{R}} \left(\frac{1}{2\pi i} \int_{\Gamma} f(z) q(z, w) \, d\rho(z) \right) \, d\mu(w) \\ &= \frac{1}{2\pi i} \int_{\Gamma} f(z) \left(\int_{\mathcal{R}} q(z, w) \, d\rho(z) \right) \, d\rho(z) \\ &= -\frac{1}{2i} \int_{\Gamma} f(z) \hat{\mu}(z) \, d\rho(z) = 0, \end{aligned}$$

so $\mu \perp R(K)$ by density. □

Lemma 2.5. *Let μ be a complex measure with compact support. If $\hat{\mu} = 0$ A-a.e. then $\mu = 0$ (the converse trivially holds).*

Proof. Let $g \in C^1_{\text{cs}}(\mathcal{R})$ (space of differentiable complex-valued functions with compact support). Then as in Lemma 2.4

$$g(w) = \frac{1}{2\pi i} \int_{\mathcal{R}} \frac{\partial g}{\partial \overline{\rho}}(z) q(z, w) \, dA(z),$$

so

$$\int_{\mathcal{R}} g(w)\ d\mu(w) = -\frac{1}{2i}\int_{\mathcal{R}} \hat{\mu}(z)\frac{\partial g}{\partial \rho}(z)\ dA(z).$$

Since $\hat{\mu} = 0$ A-a.e., we deduce that

$$\int_{\mathcal{R}} g\ d\mu = 0.$$

By the Stone-Weierstrass Theorem, $C^1_{\rm cs}(\mathcal{R})$ is dense in $C(\text{spt}\ \mu)$ so we can conclude that $\mu = 0$. □

Remark 2.6. An easy consequence of Lemma 2.5 is the Hartog-Rosenthal Theorem for Riemann surfaces [5]. Note that as the Cauchy kernel has a simple pole of degree 1 at $z = w$, for fixed w,

$$\frac{\partial}{\partial \overline{\rho}} q(z,w)\ d\overline{\rho} = \delta_w d\overline{\rho}\ ,$$

where δ_w denotes the Dirac mass concentrated at w. From this equality follows that

$$\frac{\partial}{\partial \overline{\rho}} \hat{\mu}(z)\ d\overline{\rho} = \mu(z) d\overline{\rho}\ ,$$

where these equalities are interpreted as identities between currents of bidimension (0,1).

In general the Cauchy transform of a measure μ is not continuous. However in the particular case of a measure of the form $\mu = \varphi A$ where $\varphi \in L^\infty_{\rm cs}(\mathcal{R})$ is an essentially bounded function with compact support we have the following result as a consequence of the local integrability of the Cauchy kernel.

Lemma 2.7. *Let* $\varphi \in L^\infty_{\rm cs}(\mathcal{R})$. *Then* $\widehat{\varphi A}$ *is continuous.*

Proof. Let $D := D(z,r)$ be a parametric disc of center $z \in \mathcal{R}$ and radius $r > 0$. In D, $q(z,w) = (\rho(z) - \rho(w))^{-1} + \tilde{q}(z,w)$ where $\tilde{q}(z,w)$ is analytic, so

$$\begin{aligned}\widehat{\varphi A}(w) &= \int_{\mathcal{R}} \varphi(z) q(z,w)\ dA(z)\\ &= \int_D \frac{\varphi(z)}{\rho(z)-\rho(w)}\ dA(z) + \int_D \tilde{q}(z,w)\ dA(z) + \int_{\mathcal{R}\setminus D} \varphi(z) q(z,w)\ dA(z),\end{aligned}$$

where $w \in D$. Hence

$$\widehat{\varphi A}(w) = \int_{\rho(D)} \frac{(\varphi\circ\rho^{-1})(z)}{z - \rho(w)}\ dxdy + \text{continuous function},$$

and consequently the continuity of $\widehat{\varphi\ A}$ follows from the standard argument in the complex plane. □

3. Main result

In this section we will prove that given an open subset U of $\mathcal{R}$ with essential boundary and clos U compact, then the algebra $A(U)$ is complex pervasive on bdy U. Before proving this theorem we introduce some definitions and results.

Definition 3.1. Let U be an open subset of $\mathcal{R}$ and let $a \in \text{bdy}\, U$. We say that a is an $A(U)-$*inessential boundary point* if there exist $r > 0$ such that each function $f \in A(U)$ extends analytically to $D(a, r)$.

The $A(U)$-*essential boundary* of U is the set of points in bdy U which are not $A(U)$-inessential. We abbreviate $A(U)$-essential to essential.

Remark 3.2. It is easy to show that if U has at least one inessential boundary point then $A(U)$ is not complex pervasive on bdy U. For suppose a is an inessential boundary point. Then, since

$$A(U \,\cup\, \{\text{iness. bdy. points}\}) \neq C(\text{bdy}\, U \,\cup\, \{\text{iness. bdy. points}\})$$

we can find a non-trivial annihilating measure supported on bdy $U \setminus \{a\}$.

Note also that if bdy U is essential then by Riemann's Removable Singularities Theorem, U cannot have isolated boundary points.

The next result is a classical theorem of R. Arens [1].

Theorem 3.3. *Let U be an open subset of $\mathcal{R}$, with* clos U *compact. Then the maximal ideal space* $\mathcal{M}_{A(U)} = \text{clos}\, U$.

The following theorem is folklore, and goes back to E. Bishop and L. Kodama, who studied uniform algebras of analytic functions on Riemann surfaces in the 1960's [2, 4, 12]. We include a proof for the reader's convenience. It is a pleasure to acknowledge helpful correspondence about this from T.W. Gamelin.

Theorem 3.4. *Let U be an open subset of $\mathcal{R}$, with* clos U *compact. Then the Shilov boundary of $A(U)$ is the essential boundary of U.*

Proof. By Theorem 3.3, the Shilov boundary of $A(U)$ can be viewed as a closed subset of clos U. Note that as a consequence of the maximum modulus principle for analytic functions, the essential boundary of U is contained in the Shilov boundary of $A(U)$. To prove the converse we argue by contradiction.

Suppose a is an essential boundary point of U which is not in the Shilov boundary of $A(U)$. Pick a representing measure θ for a on $A(U)$ supported on the Shilov boundary of $A(U)$, and let $\mu = (\rho(z) - \rho(a))\ \theta$, where ρ is a global uniformizing parameter on $\mathcal{R}$.

By adding a constant if necessary, we can suppose that at the point $w = a$ the Cauchy kernel $q(z, a) = (\rho(z) - \rho(a))^{-1}$, so that $\hat{\mu}(a) = 1$.

Let V the connected component of a in $\mathcal{R} \setminus \text{spt}\, \mu$. Then, as $\hat{\mu}$ is analytic off spt μ and $\hat{\mu}(a) = 1$, $\hat{\mu}$ has only a discrete set of zeros in V. Therefore the measure ν_b defined by

$$d\nu_b(z) = -\frac{1}{\pi}\frac{1}{\hat{\mu}(b)} q(z, b)\ d\mu(z) \tag{3.1}$$

is a complex representing measure for $b \in V \cap \text{clos}\, U$ on $A(U)$ [5], except perhaps for a discrete set of points, so for each $f \in A(U)$

$$f(b) = \int_{\text{clos}\ U} f(z)\ d\nu_b(z)\ .$$

Therefore each $f \in A(U)$ is holomorphic on V (by differentiation under the integral sign), except for a discrete set of singularities. Since f is bounded, f must be holomorphic on V.

Thus, $V \setminus U$ consists only of inessential boundary points of U, which is a contradiction. □

Remark 3.5. By Theorem 3.3, $\mathcal{M}_{A(U)}$ is metrizable, so there exist a minimal boundary for $A(U)$ [3], which coincides with the set of its peak points. The minimal boundary is dense in the Shilov boundary of $A(U)$. Therefore, as a consequence of Theorem 3.4, the set of peak points for $A(U)$ is dense in the essential boundary of U.

Theorem 3.6. *Let U be an open subset of $\mathcal{R}$, with* clos U *compact. Suppose that* bdy U *is essential. Then $A(U)$ is complex pervasive on* bdy U *if and only if* bdy U_i = bdy U *for each component U_i of U.*

Proof. Suppose first that U has a component U_i with bdy $U_i \neq$ bdy U. We can choose a nonzero annihilating measure μ for $A(U_i)$ supported on bdy U_i. Then $\mu \perp A(U)$ and spt μ is a proper subset of bdy U, so $A(U)$ is not complex pervasive on bdy U.

For the converse, assume that bdy U_i = bdy U for each component U_i of U. Let μ be a complex measure supported on bdy U, $\mu \perp A(U)$ and suppose that spt $\mu \neq$ bdy U.

By hypothesis, $\mu \perp A(U)$ so $\mu \perp R(\text{clos } U)$ and therefore Lemma 2.4 implies that $\hat{\mu} = 0$ off clos U.

Suppose now $a \in \text{bdy } U \setminus \text{spt } \mu$. Choose $r > 0$ sufficiently small and a parametric disc $D(a,r)$ such that clos $D(a,r) \cap \text{spt } \mu = \emptyset$. Let $\{p_n\}_{n=1}^{\infty} \subset D(a,r) \cap \text{bdy } U$ be a sequence of distinct peak points for $A(U)$ such that $p_n \longrightarrow a$ as $n \uparrow \infty$. Then there exist a sequence $f_n \in A(U)$ such that $f_n(p_n) = 1$ and $|f_n(z)| < 1$ for all $z \in (\text{clos } U) \setminus \{p_n\}$. Hence $f_n^k \longrightarrow 0$ uniformly on spt μ as $k \uparrow \infty$.

Now $\hat{\mu}(p_n) = 0$ because otherwise (note that $\hat{\mu}$ is analytic near clos $D(a,r)$) the measure ν_{p_n} defined by the equation (3.1) is a complex representing measure for p_n on $A(U)$ and

$$1 = f_n^k(p_n) = \int_{\text{bdy } U} f_n^k \, d\nu_{p_n} \longrightarrow 0 \quad \text{as } k \uparrow \infty$$

which is a contradiction.

Consequently, a is an accumulation point of zeros of $\hat{\mu}$ and by analyticity, as $\hat{\mu} = 0$ off spt μ (cf. Remark 2.3), we can conclude that $\hat{\mu} = 0$ on clos $D(a,r)$. By hypothesis bdy U_i = bdy U for each component U_i of U, so by connectivity $\hat{\mu} = 0$ on U_i, and hence $\hat{\mu} = 0$ on U. Therefore $\hat{\mu} = 0$ on $\mathcal{R} \setminus \text{spt } \mu$.

Finally, let $E \subset \text{bdy } U$ be compact. Consider the measure $\lambda = \chi_{|E} A$. By Lemma 2.7, $\hat{\lambda}$ is continuous and therefore $\hat{\lambda} \in A(U)$ so

$$0 = \int_{\text{bdy } U} \hat{\lambda} \, d\mu = -\int_{\mathcal{R}} \hat{\mu} \, d\lambda = \int_E \hat{\mu} \, dA \ ,$$

and therefore $\hat{\mu} = 0$ A-a.e. on bdy U.

Hence $\hat{\mu} = 0$ A-a.e. on $\mathcal{R}$, so by Lemma 2.5 $\mu = 0$ which proves the complex pervasiveness of $A(U)$ in bdy U. □

References

[1] R. Arens, The closed maximal ideals of algebras of functions holomorphic on a Riemann surface, *Rend. Circ. Mat. Palermo* **7** (1958), 245-260.

[2] E. Bishop, The structure of certain measures, *Duke Math. J.* **25** (1958), 283-290.

[3] ———, A minimal boundary for function algebras, *Pacific J. Math.* **9** (1959), 629-642.

[4] ———, Boundary measures of analytic differentials, *Duke Math. J.* **27** (1960), 331-340.

[5] A. Boivin, T-invariant algebras on Riemann surfaces I, *Mathematika* **34** (1987), 160-171.

[6] J. Čerych, A word on pervasive function spaces, in "Complex Analysis and Applications, Varna, 1981," pp. 107-109, Sofia, 1984.

[7] T. Gamelin, "Uniform Algebras," Prentice-Hall, Englewood Cliffs, N. J., 1969.

[8] T. Gamelin and J. Garnett, Pointwise bounded approximation and Dirichlet algebras, *J. Functional Analysis* **8** (1971), 360–404.

[9] P.M. Gauthier, Meromorphic uniform approximation on closed subsets of open Riemann surfaces, *Aproximation Theory and Functional Analysis* (Proc. Conf. Campinas 1977, J.B. Prolla ed.) 1979, 139–158.

[10] R.C. Gunning and R. Narasimhan, Inmersion of open Riemann surfaces, *Math. Ann.* **174** (1967), 103–107.

[11] K. Hoffman and I.M. Singer, Maximal algebras of continuous functions, *Acta Math.* **103** (1960), 217–241.

[12] L. Kodama, Boundary measures of analytic differentials and uniform approximation on a Riemann surface, *Pacific J. Math.* **15** (1965), 1261-1277.

[13] I. Netuka, Pervasive function spaces and the best harmonic approximation, *J. Approx. Theory* **51** (1987), 175–181.

[14] I. Netuka, A.G. O'Farrell and M.A. Sanabria-García, Pervasive algebras of analytic functions, preprint 1999.

[15] A. Sakai, Localization theorem for holomorphic approximation on open Riemann surfaces, *J. Math. Soc. Japan* **24** (1972), 189-197.

[16] S. Scheinberg, Uniform approximation by functions analytic on a Riemann surface. *Ann. Math.* **108** (1978), 257–298.

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