Research in Progress, March 2000

Anthony G. O'Farrell, Mathematics

It's never easy for a mathematician to explain to the common reader what he does. A modicum of technical knowledge is indispensible for understanding even the content of undergraduate programmes, and few attain this, even in the scientific community. Apart from its depth and sophistication, which Mathematics shares with some other areas of enquiry, there is also the unusual feature of its cumulative success. Whereas natural scientists tend to re-invent their disciplines periodically, jettisoning as baggage, or even rejecting as disproved, the theories of yesteryear, and others such as philosophers endlessly assail the same perenniel questions, mathematicians continue to treasure the achievements of the remote past, and to build on them. This makes for a formidable barrier to entry by the casual tourist.

Mathematics is tightly interconnected, each area threading into the others in manifold ways. For this reason, my research extends into many corners, depending on the needs of the moment. However, my centre of gravity lies in Mathematical Analysis. Here I use Analysis in its modern mathematical sense, which is different from its ancient mathematical sense. Analysis is that area of Mathematics (about a third of the whole) in which topological ideas predominate. In relatively lay terms, it is concerned with such concepts as limits, infinite sets and processes, approximation, continuity, differential and integral calculus, smooth functions, nonlinear processes, dynamical systems, sequences and series. In more general terms, it is concerned with the precise manipulation of ideas such as 'close to', 'shaped like', 'rapidly–changing', and concepts such as length, area, and volume. Analysis is extremely useful. Without it, most of the civilized amenities we enjoy would not exist. Archimedes had some ideas that belong to Analysis, but the subject really began in the seventeenth century, with Newton and Leibniz. It provides the bedrock for the technology we have developed since then, and also for our present understanding of the natural world — not that I wish to exaggerate this understanding. Of more interest to me is the rather unexpected fact that Analysis finds application in relation to many of the great open questions in Mathematics, including the problems that seem to lie in the other two–thirds. I will give one example.

For about 2300 years, the problem whether or not the circle could be squared was open. 'Squaring the circle' has recently become a cliché of journalists, and its original sense has been lost in the process. The problem was whether, using only straight–edge and compass, one could construct a square having the same area as a given circle. The context of Euclidean Geometry is assumed. By the early nineteenth century it was realised that the answer would be no, provided the number π were transcendental. A number x is transcendental if there are no whole numbers a_0, a_1, $\ldots$,a_n (not all zero) such that

$$a_0 + a_1 x + \cdots + a_n x^n = 0. \qquad (1)$$

This converted the problem from a problem about geometry to a problem about real numbers. Eventually, in 1882, Lindemann used methods of Analysis to show that π is indeed transcendental, so the circle cannot be squared. Hobson, in his little book on the subject (cf. [1]), remarks that perhaps the most striking aspect of this story is the monumental patience of the mathematical community (regarded as a single organism), which ground away at this problem for thousands of years until the necessary ideas were assembled to solve it.

We live in interesting times. Perhaps half the serious mathematicians who have ever lived are alive today, working conditions are generally good, and the pace of progress is breathtaking. People will probably have heard about the proof by Wiles of Fermat's Last Theorem a few years ago, and the proof of the Four-colour Theorem by Appel and Haken in the eighties. Many of the great problems extant when I was a boy have been settled, although few are intelligible to non–experts. People even talk about plans for proving the Riemann hypothesis. This problem dates from about 1850, and is not so easy to describe as the problem of squaring the circle. The formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2}$$

defines a real–valued function $\zeta(s)$ when s is a real number greater than 1. It is also possible to consider *complex* numbers s, and the series then converges to a complex number $\zeta(s)$ whenever the real part of s is greater than 1. This function $\zeta(s)$, thus *a priori* defined for some complex numbers, can be extended in a unique canonical way to all complex s except $s = 1$, even thought the formula makes no sense except when the real part of s is greater than 1. The way to extend it is as an *analytic* function. This means that the complex derivative

$$\zeta'(s) = \lim_{h\to 0, h \text{ complex}} \frac{\zeta(s+h) - \zeta(s)}{h} \tag{3}$$

exists at each s except 1. There are subtle connections between this function and the distribution of the prime whole numbers, and various questions about the factorization of whole numbers. Now it turns out that $\zeta(s) = 0$ when $s = -2, -4, -6, \ldots$, and that there are many other s with $\zeta(s) = 0$ of the form $s = \sigma + i\tau$ with σ between 0 and 1. Riemann conjectured that *all* these other 'zeros' have σ exactly equal to $\frac{1}{2}$. Most people believe this is probably true, and it has been verified for millions of the zeros, but a proof is lacking in general. Personally, I believe that a proof is still far away.

Analytic functions, described above as those having a "complex derivative", may also be described as those having local power series expansions

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a_n)^n, \qquad z \text{ near } a \tag{4}$$

or as those satisfying special integral identities

$$\int_\Gamma f(z)dz = 0 \tag{5}$$

or

$$f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz. \tag{6}$$

They have extraordinarily tight properties, not least the equivalent descriptions just mentioned. I first heard about them at the age of 18, and was immediately hooked. The power of analytic function theory is quite uncanny; one is repeatedly surprised and delighted by the amazing results and the unexpected applications. I've spent most of the last 35 years exploring them.

One can consider analytic functions $f(z_1, \ldots, z_d)$ of *several* complex variables instead of one. I am interested in questions about the iteration of such functions, their singularities, and their approximation. I am interested in questions about special families of such functions. These questions, in turn, provoke questions about structures such as Banach spaces of functions, algebras of functions, about integral estimates, capacities and Hausdorff measures, about polynomial hulls and the geometry of complex manifolds, about extension problems, derivations, harmonic functions, pseudoconvexity, etc.

Right now, Alejandro Sanabria and I are thinking about the so–called *pervasive* algebras of analytic functions, in the context of Riemann surfaces. I will try to explain the idea.

Let's start with 'algebra'. Algebra is the name of another third of Mathematics, that part concerned, roughly speaking, with the ramifications of formal manipulation of expressions. Included within its scope are many kinds of 'algebraic structure', by which I understand a set equipped with some operations. Examples are the structures known as groups, rings, fields, vector spaces, categories, and (confusingly enough) *algebras*. In other words, an algebra is a special kind of algebraic structure, and the theory of algebras is a small part of Algebra. There are many (infinitely many) algebras in this sense. Each is a set of objects, equipped with operations called addition, multiplication, and scalar multiplication, satisfying certain axioms, such as the distributive law

$$a(b+c) = ab + ac \tag{7}$$

and the associative law of multiplication

$$a(bc) = (ab)c. \tag{8}$$

For example, the set of complex numbers, equipped with its usual operations, is an algebra.

Next, take 'function'. A function on a set A is a rule that produces exactly one value $f(x)$ for each element x belonging to the set A. The value $f(x)$ may be an object of a different kind to x. If the values lie in a set B, one calls f a B–valued function. For instance, the rule $x \mapsto x^2$ takes each complex number x and produces the complex number x^2. This is an example of a complex–valued ($\mathbf{C}$–valued) function on the set $\mathbf{C}$ of complex numbers. You will recall that the set of complex numbers may be visualised as a plane (the Argand diagram, die Gaussische Ebene). This allows geometric thinking about complex numbers.

Now fix the set A and consider the set of *all* $\mathbf{C}$–valued functions on A. This set inherits an algebra structure from the algegra structure of $\mathbf{C}$. One just defines the sum and product of two functions f and g by setting

$$(f+g)(x) = f(x) + g(x),$$
$$(fg)(x) = f(x)g(x),$$

whenever x belongs to A. Thus we obtain an *algebra of functions.* It contains many smaller algebras of functions.

If all the functions in an algebra of functions are analytic, then we have an *algebra of analytic functions.* For this, the underlying set A should be $\mathbf{C}$, or a higher–dimensional complex space $\mathbf{C}^d$, or perhaps a more general object. The most general case is that in which A is a *complex manifold*, something that looks locally like $\mathbf{C}^d$, but has a more interesting global shape, rather in the way that a sphere is more interesting than a plane. A one–dimensional complex manifold is called a Riemann surface, and examples of these look like a sphere or a torus.

We are studying algebras of analytic functions on a Riemann surface.

The algebras we study have a topology. This means that we can talk about the *approximation* of one function (belonging to the algebra) by a sequence of others. Pervasiveness is related to approximation. Rather than describe the general notion, I will just give an example.

If you remove a small piece from a circle contained in the complex plane, then you can approximate any given continuous complex–valued function as closely as you like on the rest of the circle, using functions analytic inside the circle. Last year (together with Ivan Netuka from Prague) we investigated the extent to which this phenomenon depends on the particular geometry of circles, and we found that it matters very little. The disc may be replaced by any 'open, connected' set. We completed a thorough analysis of this phenomenon in the plane (cf. [2]). But the plane is the simplest Riemann surface, so currently we are examining the corresponding phenomenon on general 'open' Riemann surfaces. We are also looking at the phenomenon in another interesting context, that of certain subsets of the boundary of pseudoconvex open sets in two complex variables.

On the back burner while we pursue this work, are some other questions in several complex variables. I will not attempt to describe these here.

References

1. E.W. Hobson et al. *Squaring the Circle: and other monographs.* New York. Chelsea Publications. 1953.

2. I. Netuka, A.G. O'Farrell, and M.A. Sanabria–Garcia. *Pervasive algebras of analytic functions.* Preprint. Mathematics Dept., NUIM, 1999.