T–invariance

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Abstract.

The paper is about the function spaces on the plane that are invariant under the action of the Vitushkin localisation operator, defined by

$$T_\phi f = \left(\frac{-1}{\pi z}\right) * \left(\phi \cdot \bar{\partial} f\right)$$

for $\phi \in {C^\infty}_{\mathrm{cs}}$ and suitable distributions f. Such spaces are called T–invariant. The question of T–invariance is examined in the context of translation–symmetric concrete spaces (TSCS). Roughly speaking, these are complete locally–convex topological vector spaces of distributions that contain the test functions, are modules under multiplication by test functions, and are closed under complex conjugation and translation. The main result is this:

Theorem. *Suppose F is a TSCS that admits a separable translation–measurable TSCS topology. Then F is locally T–invariant, and F_∞ is T–invariant.*

The most useful consequence is:

Corollary. *Let (1) F be a small TSCS or (2) F be the TSCS dual of a small TSCS. Then F is locally T–invariant, and F_∞ is T–invariant.*

A small TSCS is one in which ${C^\infty}_{\mathrm{cs}}$ is sequentially–dense. The space F_∞ is locally–equivalent to F, and is constructed from F in a canonical way.

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1. Introduction.

Let $C^\infty(\mathbb{C})$ denote the Frechet space of infinitely–differentiable functions $f:\mathbb{C}\to\mathbb{C}$, and let ${C^\infty}_{\rm cs}$ denote the nuclear space of all functions $f\in C^\infty$ that have compact support. Consider the space of complex–valued distributions on the complex plane, the dual ${{C^\infty}_{\rm cs}}'$ of ${C^\infty}_{\rm cs}$. The *Cauchy transform* is the convolution operator $\mathfrak{C}$, defined on certain distributions f by

$$\mathfrak{C}f=\left(\frac{-1}{\pi z}\right)*f.$$

More precisely, $\mathfrak{C}$ is defined in this way on the space $C^{\infty\prime}$ of all distributions having compact support, and it can be extended continuously to many larger topological vector spaces of distributions, in which $C^{\infty\prime}$ is dense. The Cauchy transform inverts the $\bar\partial$ operator

$$\bar\partial f=\frac{\partial f}{\partial\bar z}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+\iota\frac{\partial f}{\partial y}\right)$$

on the distributions having compact support.

The *Vitushkin localisation operator* is defined by

$$\begin{aligned}T_\phi f&=\mathfrak{C}\left(\phi\cdot\bar\partial f\right)\\&=\phi\cdot f-\mathfrak{C}\left((\bar\partial\phi)\cdot f\right)\end{aligned}$$

for $\phi\in {C^\infty}_{\rm cs}$ and suitable distributions f. The purpose of this paper is to discuss the continuity properties of this operator, which has played a crucial role in connection with a number of investigations in complex analysis. Vitushkin himself used it in his penetrating study of uniform rational approximation [10]. Arens [1] used it to show that the maximal ideal space of the uniform algebra of all functions continuous on a compact set $X\subset\mathbb{C}$ and analytic on int$X$ is X. Gamelin and Garnett [4, 5] used it in their work on bounded analytic functions. Davie [2] used it to study the bounded and continuous analytic capacities. The author [6, 7] and others used it in connection with holomorphic approximation problems in Lipschitz and other norms. The utility of the operator is explicable in terms of the equation

$$\bar\partial T_\phi f=\phi\cdot\bar\partial f.$$

This shows that $T_\phi f$ is analytic wherever f is and off sptϕ, and that $f-T_\phi f$ is analytic on the interior of $\phi^{-1}(1)$. Thus it may be used to split up the set of essential singularities of f. The idea behind this goes back to the derivation of the Laurent series, based on the splitting of a function f, analytic between two circles into a function f_1, analytic inside the outside circle, plus a function f_2, analytic outside the inside circle. The usual construction of f_1 and f_2 uses line integrals. Line integrals have more restricted domains than area integrals, so one is led to use the other term of Pompeiu's formula, replacing the line integrals by area integrals. The result is the Vitushkin localisation operator, with a special ϕ.

In some situations, one meets the operator in the variant form

$$T_\phi^X f(w)=\frac{1}{\pi}\int\int_X\frac{f(z)-f(w)}{z-w}\frac{\partial\phi}{\partial\bar z}dxdy,$$

where X is a closed subset of $\mathbb{C}$ and $f : X \to \mathbb{C}$ is a Lebesgue–measurable function. The study of this variant reduces to the study of the global $T_\phi (= T_\phi^{\mathbb{C}})$ when, as is often the case, there is available a suitable extension operator.

When the $\bar\partial$ operator on $\mathbb{C}$ is replaced by an elliptic operator L on $\mathbb{R}^d$, having a parametrix E, then the equivalent to the T_ϕ operator is the operator

$$T_\phi^L f = E(\phi \cdot Lf),$$

defined on certain distributions $f \in C^\infty(\mathbb{R}^d, \mathbb{C})_{\mathrm{cs}}{}'$. We will also discuss the continuity of these operators.

It has been found over the years that the T_ϕ operator acts continuously on a great variety of function spaces. This was verified in an *ad hoc* manner for each space as it came up. The main point of the present paper is that there is a simple uniform way to derive most of these results. We shall work with the class of translation–symmetric concrete spaces (TSCS) and show that a very broad family of these spaces have the property of local T–invariance.

In section 2, we introduce the TSCS and define some related concepts, including small TSCS. In section 3, we lay out some constructions that start from a TSCS, F, and produce spaces F_X, $F(X)$, F_{loc}, F_{cs}, F_∞. We establish basic properties of these spaces. In section 4, we study the convolution as a map from $F \times \mathrm{L}^1{}_{\mathrm{loc}}$ to distributions, and we establish a result (Theorem 4.1) which shows that under very general conditions, convolution maps

$$F_{\mathrm{cs}} \times \mathrm{L}^1{}_{\mathrm{loc}} \to F_{\mathrm{loc}},$$
$$F_{\mathrm{cs}} \times \mathrm{L}^1{}_{\infty} \to F_{\infty},$$

and, rather less generally,

$$F \times \mathrm{L}^1 \to F.$$

We also establish an automatic continuity result for such maps (Props. 4.3 and 4.4). In section 5, we apply these results to the T_ϕ operator, and we prove the main result:

Theorem 5.5. *Suppose F is a TSCS that admits a separable translation–measurable TSCS topology. Then F is locally T–invariant, and F_∞ is T–invariant.*

The most useful consequence is:

Corollary 5.6. *Let (1) F be a small TSCS or (2) F be the TSCS dual of a small TSCS. Then F is locally T–invariant, and F_∞ is T–invariant.*

This covers most interesting spaces. We also discuss T–invariance results for more general spaces, and similar results for other T_ϕ–like operators, associated to operators other than $\bar\partial$.

A word about notation: The special notation $X \hookrightarrow Y$ means that the topological space X is a subset of the topological space Y, and that the inclusion map is continuous. It does not mean that X has the relative topology from Y.

The *dual* F^* of a topological vector space F is the space of all continuous linear functions $T : F \to \mathbb{C}$. For our purposes, there are two interesting topologies to give F^*.

The *strong* topology is that in which a neighbourhood base at the origin is provided by the sets

$$\text{polar}(B) =_{def} \{T \in F^* : |Tf| \leq 1, \forall f \in B\},$$

where B runs over the bounded subsets of F. The *weak-star* topology is defined by the sub-base

$$\text{polar}(\{f\}) = \{T \in F^* : |Tf| \leq 1\},$$

where f runs over F. We use the notation F^* to denote F^* with the strong topology, and F' to denote the same space with the weak-star topology.

The *algebraic dual* $F^\dagger$ of F is the vector space of all linear functions (continuous or not) from F to $\mathbb{C}$.

We denote by i the natural map $i : F \to F^{*\dagger}$ given by

$$(if)(T) = Tf \quad , \quad \forall T \in F^* \; \forall f \in F.$$

We will use the following standard theorem which tells us how to distinguish the elements of iF from the other elements of $F^{*\dagger}$.

The Banach-Grothendieck Theorem. *Let F be a complete LCTVS and let $u \in F^{*\dagger}$. Then the following three conditions are equivalent:*
(1)$u \in F$, i.e. $u \in iF$;
(2)$u|\text{polar}(N)$ is weak-star continuous, whenever N is a neighbourhood of 0 in F;
(3)For some neighbourhood base B for 0 in F, we have that $u|\text{polar}(N)$ is weak-star continuous, whenever $N \in B$.

2. Translation–symmetric Concrete Spaces.

In [8] we introduced the class of symmetric concrete spaces (SCS), which admit, among other symmetries, the compositional action of the full affine group. The results of the present paper do not require the hypothesis of full affine–invariance, so we define the larger class of translation–symmetric concrete spaces. A translation–symmetric concrete space (TSCS) on $\mathbb{R}^d$ is a complete locally–convex topological vector space (LCTVS), F, such that

1. ${C^\infty}_{\mathrm{cs}} \hookrightarrow F \hookrightarrow {{C^\infty}_{\mathrm{cs}}}'$,

2. $f \mapsto \bar{f}$ maps $F \to F$ continuously,

3. $$\left\{ \begin{array}{r} {C^\infty}_{\mathrm{cs}} \times F \to F \\ (\phi, f) \mapsto \phi \cdot f \end{array} \right\}$$

makes F a topological ${C^\infty}_{\mathrm{cs}}$–module.

4. for each $T \in$Tran,

$$c_T : \left\{ \begin{array}{l} F \to F \\ f \mapsto f \circ T \end{array} \right\} \text{ is continuous}$$

and $T \mapsto c_T$ maps compact subsets of Tran to equicontinuous subsets of End(F).

Here Tran $=$ Tran$(\mathbb{R}^d)$ denotes the group of translations. For $T \in$ Tran and $f \in {{C^\infty}_{\mathrm{cs}}}'$, the 'composition' $f \circ T$ is defined by the formula

$$\langle \phi, f \circ T \rangle = \langle \phi \circ T^{-1}, f \rangle \quad , \quad \forall \phi \in {C^\infty}_{\mathrm{cs}}.$$

The map $c_T : f \mapsto f \circ T$ is a continuous linear automorphism of ${{C^\infty}_{\mathrm{cs}}}'$.

If a TSCS is normable, we call it a translation–symmetric concrete Banach space (TSCBS). If F is a metrisable TSCS, we call it a symmetric concrete Frechet space (TSCFS). The TSCBS are the most important TSCS. The others, including ${C^\infty}_{\mathrm{cs}}$, ${{C^\infty}_{\mathrm{cs}}}'$, C^∞, $C^{\infty\prime}$, the space of rapidly–decaying functions and its dual, the space of tempered distributions, some weak-star duals of Banach spaces, and the spaces F_{cs} and F_{loc} (constructed below), are merely auxilliary.

In case F is a TSCBS, the four axioms take the following form.

1. The statement ${C^\infty}_{\mathrm{cs}} \hookrightarrow F$ means that ${C^\infty}_{\mathrm{cs}} \subset F$ and that there exist continuous functions $\rho_k : \mathbb{R}^d \to [0, +\infty)$, $(k = 0, 1, 2, ...)$ such that, given $R > 0$, all but a finite number of the ρ_k vanish identically on $\mathbb{B}(0, R)$, and such that

$$\|\phi\|_F \leq \sum_{k=0}^{+\infty} \sup\{\rho_k(x) \cdot |D^k \phi(x)| : x \in \mathbb{R}^d\} \quad , \quad \forall \phi \in {C^\infty}_{\mathrm{cs}}.$$

The statement $F \hookrightarrow {{C^\infty}_{\mathrm{cs}}}'$ means that $F \subset {{C^\infty}_{\mathrm{cs}}}'$ and that, given $\phi \in {C^\infty}_{\mathrm{cs}}$, there exists $\kappa_1(\phi) > 0$ such that

$$|\langle \phi, f \rangle| \leq \kappa_1(\phi) \|f\|_F \quad , \quad \forall f \in F.$$

That $f \mapsto \bar{f}$ is bicontinuous means that for some constant $\kappa_2 > 1$ we have

$$\kappa_2^{-1} \cdot \|\bar{f}\|_F \leq \|f\|_F \leq \kappa_2 \cdot \|\bar{f}\|_F \quad , \quad \forall f \in F.$$

Thus we obtain an equivalent norm

$$f \mapsto \tfrac{1}{2}\{\|f\|_F + \|\bar{f}\|_F\}$$

on F for which $f \mapsto \bar{f}$ is an isometry.

That F is a topological ${C^\infty}_{\text{cs}}$–module now means that there exist functions ρ_k', like the ρ_k above, such that

$$\|\phi f\|_F \leq \|f\|_F \cdot \sum_{k=0}^{+\infty} \sup \rho_k' \cdot |D^k \phi| \quad , \quad \forall f \in F \quad , \quad \forall \phi \in {C^\infty}_{\text{cs}}.$$

In fact, we may take it, if we want to, that $\rho_k = \rho_k'$.

The equicontinuity statement of Axiom 4 just says that for $K \subset$Tran, K compact, we have

$$\sup_{T \in K} \|c_T\| < +\infty,$$

where $||c_T||$ stands for the operator norm of c_T on F. A word of caution is in order: Axiom 4 does not say or imply that 'translation is continuous' on F, i.e. that $f \circ T$ varies continuously in F as T varies continuously in Tran. In fact, for reasonable spaces, the continuity of the map $T \mapsto c_T$ from Tran into the endomorphisms of F implies Axiom 4, but the converse is false. With $F = L^\infty$, each c_T is an isometry, but translation is discontinuous.

Definition. An TSCS is called *small* if ${C^\infty}_{\text{cs}}$ is sequentially dense in it.

As examples, L^p, C^k, Lipα, lipα, BMO, VMO, Sobolev spaces, Besov spaces, Bloch space, Zygmund class (ZC) and Zygmund smooth class (ZS) are TSCS. The space L^p is small if $p < +\infty$. Other small spaces are C^k, lipα, $\mathrm{VMO}_{\text{loc}}$, and ZS_{loc}.

3. Some TSCS constructions.

We summarise the relevant points from TSCS theory. These parallel the corresponding facts from SCS theory, and the reader who would like to see more details could consult [9].

Let F be a TSCS. To a closed subset $X \subset \mathbb{R}^d$, we associate spaces $F(X)$ — germs on X, F_X — elements of F that are supported on X, and in terms of these we topologise the spaces

$$F_{\mathrm{loc}} = C^\infty \cdot F \qquad \text{and} \qquad F_{\mathrm{cs}} = {C^\infty}_{\mathrm{cs}} \cdot F.$$

The definitions are as follows.

For $E \subset \mathbb{R}^d$, we define

$$F_E = \{f \in F : \mathrm{spt} f \subset E\}.$$

This F_E is a vector subspace of F, and is closed in F whenever E is closed in $\mathbb{R}^d$. For compact $X \subset \mathbb{R}^d$, we define

$$J(F,X) = \mathrm{clos}_F(F_{\mathbb{R}^d \sim X}),$$
$$F(X) = F/J(F,X),$$

and we give $F(X)$ the quotient topology, so that it becomes a complete LCTVS. If F is Banach, then so is $F(X)$ with the norm

$$\|f + J(F,X)\|_{F(X)} = \inf\{\|g\|_F : g - f \in J(F,X)\}.$$

We use the notation

$$f|X = f + J(F,X),$$
$$\|f\|_{F(X)} = \|f|X\|_{F(X)},$$

for $f \in F$,and we call the map

$$\begin{cases} f \mapsto f|X \\ F \twoheadrightarrow F(X) \end{cases}$$

restriction to X. Observe that in fact

$$\|f\|_{F(X)} = \inf\{\|g\|_F : g = f \text{ near } X\}.$$

As an example,

$$\mathrm{L}^p_E = \{f \in \mathrm{L}^p : f = 0 \ a.e. \text{ off a compact subset of } E\}$$
$$J(\mathrm{L}^p, X) = \{f \in \mathrm{L}^p : f = 0 \ a.e. \text{ on } X\}$$

and $\mathrm{L}^p(X)$, as defined here, namely

$$\mathrm{L}^p/J(\mathrm{L}^p, X),$$

is isometrically isomorphic to $\mathrm{L}^p(X, m^d|X)$.

If $E_1 \subset E_2 \subset \mathbb{R}^d$, we have the inclusion $F_{E_1} \subset F_{E_2}$. Thus for compacta $X_1 \subset X_2$, we get continuous maps

$$J(F, X_2) \hookrightarrow J(F, X_1),$$
$$F(X_2) \twoheadrightarrow F(X_1).$$

We call the latter map a restriction also. This is consistent, because the diagram of restrictions

$$\begin{array}{ccc} & F & \\ \swarrow & & \searrow \\ F(X_2) & \rightarrow & F(X_1) \end{array}$$

commutes. In the Banach space case, these restrictions are contractions.

We define

$$F_{\mathrm{loc}} = C^\infty \cdot F = \{f \in {C^\infty}_{\mathrm{cs}}{}' : \phi f \in F,\ \forall \phi \in {C^\infty}_{\mathrm{cs}}\},$$
$$F_{\mathrm{cs}} = {C^\infty}_{\mathrm{cs}} \cdot F = C^{\infty\prime} \cap F.$$

The 'loc' stands for 'local' and the 'cs' for 'compact support'. These spaces inherit natural topologies which make them symmetric concrete spaces. The topology of F_{loc} is the locally–convex projective limit topology induced by the identification

$$F_{\mathrm{loc}} = \overleftarrow{\lim_{X \mathrm{compact}}} F(X).$$

In other words, each restriction

$$F_{\mathrm{loc}} \twoheadrightarrow F(X)$$

is continuous, and the topology is the minimal locally-convex topology with this property. If F is a Banach space, then $F_{\mathrm{loc}}$is a Frechet space, with topology defined by the seminorms

$$f \mapsto \|f\|_{F(X)}, \quad X \text{ compact}.$$

The topology on

$$F_{\mathrm{cs}} = \bigcup_{X \text{ compact}} F_X$$

is the locally–convex inductive limit topology, i.e. each of the inclusions $F_X \hookrightarrow F_{\mathrm{cs}}$ is continuous and the topology is the largest locally-convex topology with this property. In other words, a convex set $G \subset F_{\mathrm{cs}}$ is open if and only if $G \cap F_X$is open in F_X, for each compact $X \subset \mathbb{R}^d$. In terms of seminorms, the topology of F_{cs} may be defined by the seminorms of the form

$$f \mapsto \sum_{k=1}^{+\infty} s_k(f) \cdot \sup_{x \in \mathbb{R}^d} \rho_k(x),$$

where $s_k (k = 1, 2, 3, ...)$ are continuous seminorms on F and $\rho_k : \mathbb{R}^d \to [0, \infty)$ are continuous functions such that on each compact set all but a finite number vanish.

Happily, ${C^\infty}_{\mathrm{cs}} = (C^\infty)_{\mathrm{cs}}$.

Proposition 3.1. *If F is a TSCS, then so are F_{loc} and F_{cs}.*

Proof. We give the details for F_{loc}. The other is similar.

Each Cauchy net $\{f_\alpha\} \subset F_{\mathrm{loc}}$ restricts to a Cauchy net in each $F(X)$,hence converges in each $F(X)$. The limits are consistent under restriction, hence define an element $f \in F_{\mathrm{loc}}$, and $f_\alpha \to f$ in F_{loc} (because this just says that $f_\alpha|X \to f|X$ for each compact X). Thus $F_{\mathrm{loc}}$is complete.

Axioms 1 and 2 are simple to check.

To prove Axiom 3, we must show the continuity of the map

$$C^\infty{}_{\mathrm{cs}} \times F_{\mathrm{loc}} \to F_{\mathrm{loc}},$$
$$(\phi, f) \mapsto \phi f.$$

Take a seminorm $t : F_{\mathrm{loc}} \to [0, \infty)$, induced by restricting a seminorm $s : F \to [0, \infty)$to a compact set X, i.e.

$$t(f) = \inf\{s(g) : g \in F,\ g = f \text{ near } X\}.$$

Since $C^\infty{}_{\mathrm{cs}} \times F \to F$ is continuous there is a seminorm $u : F \to [0, \infty)$ and a collection of continuous functions $\rho_k : \mathbb{R}^d \to [0, \infty)$, such that on each compact $K \subset \mathbb{R}^d$, all but a finite number of ρ_k vanish, and such that

$$s(\phi f) \leq u(f) \cdot \sum_{k=0}^{+\infty} \sup_{x \in \mathbb{R}^d} \rho_k(x) |D^k \phi(x)| \quad , \quad \forall \phi \in C^\infty{}_{\mathrm{cs}} \quad , \quad \forall f \in F.$$

Take a function $\chi \in C^\infty{}_{\mathrm{cs}}$ such that $\chi = 1$ near X and let $v : F_{\mathrm{loc}} \to [0, \infty)$ be the seminorm induced by restricting u to $\mathrm{spt}\chi$. Then

$$t(\phi f) = t(\chi \phi f) \leq v(f) \cdot \sum_{k=0}^{+\infty} \sup_{\mathbb{R}^d} \rho_k |D^k(\chi\phi)|$$

whenever $\phi \in C^\infty{}_{\mathrm{cs}}$ and $f \in F_{\mathrm{loc}}$, and the map

$$\phi \mapsto \sum_{k=0}^{+\infty} \sup \rho_k |D^k(\chi\phi)|$$

is a continuous seminorm on $C^\infty{}_{\mathrm{cs}}$. Thus the map $C^\infty{}_{\mathrm{cs}} \times F_{\mathrm{loc}} \to F_{\mathrm{loc}}$ is continuous.

The proof of Axiom 4 is broadly similar, and we omit the details. ∎

The most useful equivalence relation on TSCS is *local equivalence*, defined by

$$F_1 \overset{\mathrm{loc}}{=} F_2 \Leftrightarrow F_{1\mathrm{loc}} = F_{2\mathrm{loc}}.$$

The notion of *local (continuous) inclusion*, defined by

$$F_1 \overset{\mathrm{loc}}{\hookrightarrow} F_2 \Leftrightarrow F_{1\mathrm{loc}} \hookrightarrow F_{2\mathrm{loc}},$$

gives a partial order on the local equivalence classes. It turns out that for Frechet TSCS the continuity of inclusion maps is automatic. In other words, for metrisable TSCS, $F_1 \subset F_2$ is equivalent to $F_1 \hookrightarrow F_2$, so that

$$F_1 \overset{\text{loc}}{\hookrightarrow} F_2 \Leftrightarrow F_{1\text{loc}} \subset F_{2\text{loc}}.$$

The usefulness of the notion of local inclusion is illustrated by the L^p spaces. One never has $\mathrm{L}^p \hookrightarrow \mathrm{L}^q$ if $p \neq q$, but $\mathrm{L}^p \overset{\text{loc}}{\hookrightarrow} \mathrm{L}^q$ if and only if $p \geq q$, so $\overset{\text{loc}}{\hookrightarrow}$ gives a linear order on the L^p spaces. In general, $\overset{\text{loc}}{\hookrightarrow}$ is not a total order on the TSCS. For instance, in two dimensions, the space $C = C^0$ of continuous functions and the Sobolev space $W^{1,2}$ are unrelated by $\overset{\text{loc}}{\hookrightarrow}$

The following "F_∞ construction" is useful:

The space F_∞ associated to an TSCS, F, is the set of all those $f \in F_{\text{loc}}$ such that

$$f(\cdot + a)|\mathbb{B}(0,1) \to 0$$

in $F(\mathbb{B}(0,1))$–topology as $a \to \infty$.

Given $F \in$TSCS, let F_∞ denote the space of all those $f \in F_{\text{loc}}$, such that

$$f(\cdot + a)|\mathbb{B}(0,1) \to 0$$

in the topology of $F(\mathbb{B}(0,1))$, as $a \to \infty$. Here $f(\cdot + a)$ denotes the composition of f with the translation $x \mapsto x + a$, i.e.

$$\langle \phi, f(\cdot + a)\rangle = \langle x \mapsto \phi(x - a), f\rangle \quad , \quad \forall \phi \in {C^\infty}_{\text{cs}}.$$

The topology of F_∞ is defined by the seminorms u obtained as follows. Take a seminorm $s : F \to [0, +\infty)$. Let $t : F(\mathbb{B}(0,1)) \to [0, +\infty)$ be the induced seminorm, given by

$$t(h) = \inf\{s(f) : f|\mathbb{B}(0,1) = h\}.$$

The seminorm $u : F_\infty \to [0, +\infty)$ is defined by

$$u(f) = \sup\{t(f(\cdot + a)) : a \in \mathbb{R}^d\}.$$

If F is a Banach space, then so is F_∞, and the norm is given by

$$\|f\|_{F_\infty} = \sup\{\|f(\cdot + a)\|_{F(\mathbb{B}(0,1))} : a \in \mathbb{R}^d\}.$$

In general, F_∞ is a new TSCS, and is locally equivalent to F.

As an example, L^p_∞ is the space of those $f \in {\mathrm{L}^p}_{\text{loc}}$ such that

$$\|f\|_{\mathrm{L}^p(\mathbb{B}(a,1))} \to 0 \text{ as } a \to \infty,$$

and its norm is given by

$$\|f\|_{\mathrm{L}^p_\infty} = \sup_{a \in \mathbb{R}^d} \|f\|_{\mathrm{L}^p(\mathbb{B}(a,1))}.$$

Observe that the topology of F_∞ is defined by a family of translation-invariant seminorms. Thus the family of all translations is equicontinuous on F_∞. If F is a Banach space, then each translation is an isometry on F_∞.

The space F_∞ may be larger or smaller than the original F. For instance, C^0_∞ is the space, often denoted C_0, of continuous functions that tend to zero at infinity, and is smaller than C^0, whereas ${\rm L}^2{}_\infty$ is the space of measurable functions that have

$$\int_{\mathbb{B}(a,1)} |f|^2 dx \to 0$$

as $a \to \infty$, and is larger than ${\rm L}^2$.

For any TSCS, F, there is a canonical map $i : F^* \to {C^\infty}_{\rm cs}{}'$, the adjoint of the inclusion map ${C^\infty}_{\rm cs} \hookrightarrow F$. This map is injective if and only if ${C^\infty}_{\rm cs}$ is dense in F (by the Hahn-Banach theorem). Once that happens, both F^* (the strong dual) and F' (the weak-star dual) are isomorphic to symmetric concrete spaces, and we refer to the image of F^* in ${C^\infty}_{\rm cs}{}'$ as the concrete dual of F. For instance, ${\rm L}^q$ is the concrete dual of ${\rm L}^p$ when $1 \le p < +\infty$, $q = p/(p-1)$.

In the opposite direction, if a TSCS, F, is isomorphic to the dual of some LCTVS, G, then G itself is isomorphic to some TSCS if and only if ${C^\infty}_{\rm cs}$ is weak-star dense in F. Indeed, this follows on applying the remark of the last paragraph to (F, weak-star).

When a TSCBS is a concrete dual space F^*, it is occasionally useful to pass to F' to get results about F^*. For instance, F' is separable, whereas F^* may not be. The most important examples are the spaces ${\rm L}^\infty$ and Lipα (see below).

Let F be an TSCS in which ${C^\infty}_{\rm cs}$ is dense. For closed $X \subset \mathbb{R}^d$, consider the spaces:

$F(X)^*$: the dual of the restriction space $F(X)$,
$(F^*)_X$: the space of elements of the dual F^* having support in X,
$F^*(X)$: the restriction of the dual space, and
$(F_X)^*$: the dual of the space of elements of F that are supported on X.

There is a duality between these restriction spaces and support spaces, given by the following.

Proposition 3.2. *If ${C^\infty}_{\rm cs}$ is dense in the TSCS F and X is a closed subset of $\mathbb{R}^d$, then*

$$F(X)^* = (F^*)_X, \qquad (F_X)^* = F^*(X).$$

(Strictly speaking these equalities are natural isomorphisms.)

Proof. We give the details for the first part. The other part is proved in a similar way.

The map

$$F(X)^* \to F^*$$
$$T \mapsto g$$

where $\langle f, g\rangle = \langle f|X, T\rangle$, $\forall f \in F$, sends $F(X)^*$ injectively and continuously into $(F^*)_X$. To see that it is onto, it suffices to fix $g \in (F^*)_X$ and show that g annihilates $J(F, X)$. For this, it suffices to show that $\langle f, g\rangle = 0$ whenever $f \in F$ has support disjoint from X. Fix such an f, and let $\psi \in {C^\infty}_{\rm cs}$ be such that $\psi = 1$ near spt$f$. Take a net $\{\phi_\alpha\}$, with $\phi_\alpha \to f$ in F. By the module property of F, $\psi\phi_\alpha \to \psi f$ in F topology, hence $(1-\psi)\phi_\alpha \to (1-\psi)f = f$, whence $0 = \langle (1-\psi)\phi_\alpha, g\rangle \to \langle f, g\rangle$. We conclude that

$$F(X)^* \to (F^*)_X$$

is an isomorphism. ∎

In the event that the space F is an SCS, then so are the spaces F_{loc}, F_{cs}, and F_∞.

4. Convolution on TSCS.

We now consider the action of convolution by a locally–integrable function on elements of a translation–symmetric concrete space.

For Lebesgue measurable functions $f, g : \mathbb{R}^d \to \mathbb{C}$, the *convolution* $(f * g)(x)$ is defined by

$$(f * g)(x) = \int f(x-y)g(y)\,dy$$

whenever $f(x - \cdot)g \in \mathrm{L}^1$.

Convolution may be extended to various kinds of distributions, by starting from the observation that

$$\int (f * g)(x)h(x)dx = \int\int f(z)g(y)h(y+z)\,dy\,dz$$

whenever $f, g, h \in {C^\infty}_{\text{cs}}S$. this formula suggests the definition

$$\langle \phi, f * g\rangle = \langle z \mapsto \langle y \mapsto \phi(y+z), f\rangle, g\rangle \quad , \quad \forall \phi \in {C^\infty}_{\text{cs}}$$

for the convolution of distributions f and g. This makes sense, for instance, when one of f, g has compact support. Thus we may consider it for $f \in F_{\text{cs}}$ and $g \in {\mathrm{L}^1}_{\text{loc}}$. If the map $f \mapsto f * g$ is F_1–topology to F_2–topology continuous on $F_{1\text{cs}}$, then it extends to a unique continuous map of F_1 into F_2. We consider when this might occur, with particular reference to the cases when F_1 and F_2 are locally–equivalent. There is quite a variety of possible results. We will give three in the following proposition, but first we need a definition.

Definition. Let F be a TSCS. We say that *translation is weakly measurable* on F if each of the functions

$$y \mapsto \langle \tau_y f, h\rangle,$$

is Lebesgue measurable, where $h \in F^*$, $f \in F$, and $\tau_y f$ denotes the translation of f by $-y$:

$$\langle \psi, \tau_y f\rangle = \langle x \mapsto \psi(x+y), f\rangle \quad , \quad \forall \psi \in {C^\infty}_{\text{cs}}.$$

For instance, translation is weakly measurable on L^p, $p < +\infty$, on $\mathcal{C}^k$, and on (L^∞, weak–star). We will shortly see other cases.

Theorem 4.1. *Suppose an TSCS, F, is separable and that translation is weakly measurable on F. Then:*

(1) Convolution maps $F_{\mathrm{cs}} \times {\mathrm{L}^1}_{\mathrm{loc}} \to F_{\mathrm{loc}}$, continuously.
(2) Convolution maps $F_{\mathrm{cs}} \times \mathrm{L}^1_{\infty} \to F_{\infty}$, continuously.
(3) Suppose in addition that all translations act equicontinuously on F. Then convolution maps $F \times \mathrm{L}^1 \to F$, continuously.

Remark. Part (2) looks somewhat arcane, but is useful because some interesting convolution kernels like the Cauchy kernel in $\mathbb{C}$ and the Newtonian kernel in $\mathbb{R}^d$ ($d > 2$) belong to ${\mathrm{L}^1}_{\infty}$.

Proof. We give the details for (1). Let $g \in {\mathrm{L}^1}_{\infty}$, and $f \in F_{\mathrm{cs}}$. We begin by extending the domain of $f * g$ from ${C^\infty}_{\mathrm{cs}}$ to $(F_{\mathrm{loc}})^* = (F^*)_{\mathrm{cs}}$, i.e. we define $f * g$ as a linear functional from $(F_{\mathrm{loc}})^*$ to $\mathbb{C}$, by setting

$$\langle h, f * g \rangle = \int \langle h, \tau_y f \rangle g(y)\, dy \quad , \quad \forall h \in (F_{\mathrm{loc}})^*.$$

The Lebesgue measurable function $u : y \mapsto \langle h, \tau_y f \rangle$ is bounded, and vanishes off a compact set, because spt$f$ is compact and h is continuous on some $F(X)$, X compact. Thus $f * g$ is a well- defined linear functional on $(F_{\mathrm{loc}})^*$. To show it actually belongs to F_{loc}, it suffices, by the Banach-Grothendieck theorem, to show that $f * g$ is weak–star continuous on equicontinuous subsets of $(F_{\mathrm{loc}})^*$. Using the separability of F_{loc}, it reduces to showing that if a bounded sequence $\{h_n\}_{n=1}^{+\infty} \subset (F_{\mathrm{loc}})^*$ converges weak-star to $h \in (F_{\mathrm{loc}})^*$, then

$$\langle h_n, f * g \rangle \to \langle h, f * g \rangle.$$

For each $y \in \mathbb{R}^d$, we have

$$\langle h_n, f * g \rangle \to \langle h, \tau_y f \rangle.$$

Since $\{h_n\}_{n=1}^{+\infty}$ is bounded, there is a seminorm s on F_{loc} such that $\{h_n\}_{n=1}^{+\infty} \subset$ polar$\{s \leq 1\}$, i.e.

$$|\langle h_n, \tau_y f \rangle| \leq s(\tau_y f) \quad , \quad \forall y \in \mathbb{R}^d \quad , \quad \forall n \in \mathbb{N}.$$

Since $s(g) = 0$ when spt$g$ is outside a certain compact, and spt$f$ is compact, axiom 4 yields a constant $M > 0$ such that

$$s(\tau_y f) \leq M \quad , \quad \forall y \in \mathbb{R}^d.$$

Thus the Lebesgue dominated convergence theorem yields

$$\int \langle h_n, \tau_y f \rangle g(y)\, dy \to \int \langle h, \tau_y f \rangle g(y)\, dy,$$

which is what we want. ∎

The experts will recognise the foregoing as a variant of a standard result on vector integration ([3], Theorem (8.14.14) p. 570). One might remark that the intuition behind this result is quite simple. Convolving a distribution with a locally–integrable function is a process of taking limits of averages over translates of the distribution. TSCS are nicely preserved under translation, and they are complete, so that it is reasonable to suppose that they will be essentially preserved by convolutions.

Definition. We say that F is *countably sequentially generated by* ${C^\infty}_{\mathrm{cs}}$ if ${C^\infty}_{\mathrm{cs}}$ is dense in F in the following special way: Let F_1 be the sequential closure of ${C^\infty}_{\mathrm{cs}}$ in F. Let $F_{\omega+1}$ be the sequential closure of F_ω. For limit ordinals ω, let F_ω be the union of the F_β, $\beta < \omega$. The assumption is that $F = F_{\aleph_1}$.

For metrisable TSCS, F is countably sequentially generated by ${C^\infty}_{\mathrm{cs}}$ if and only if ${C^\infty}_{\mathrm{cs}}$ is dense in F.

Lemma 4.2. *Suppose F is a TSCS and is countably sequentially generated by ${C^\infty}_{\mathrm{cs}}$. Then F is separable and translation is weakly–measurable on F.*

Proof. The space ${C^\infty}_{\mathrm{cs}}$ is separable, and is F–dense in F, hence any countable ${C^\infty}_{\mathrm{cs}}$–dense subset of ${C^\infty}_{\mathrm{cs}}$ is F–dense in F. Consequently, F^* is a space of distributions.

For $\phi \in {C^\infty}_{\mathrm{cs}}$ and $g \in F^*$, the function

$$x \mapsto \langle (y \mapsto \phi(x+y)), g\rangle$$

is continuous, and it follows from the hypothesis that for $f \in F$, the function

$$x \mapsto \langle \tau_x f, h\rangle$$

is a Baire function, and hence is Lebesgue measurable. ∎

The main consequence is that Theorem 4.1 applies in the cases when:

(1) F is any small TSCS, and in particular any Banach space in which ${C^\infty}_{\mathrm{cs}}$ is dense,
(2) F is the weak-star dual G' of any small TSCS, G.

An important point to note is that even in the second case we get a *strong* continuity result for $f * g$. We formalise this, first in the Banach space case.

Proposition 4.3. *Let F be a TSCBS (separable or not).*

(1) Suppose only that convolution maps $F_{\mathrm{cs}} \times {L^1}_{\mathrm{loc}} \to F_{\mathrm{loc}}$. Then for each compact $X \subset \mathbb{R}^d$, there exists a compact set $Y(X) \subset \mathbb{R}^d$ and a constant $\kappa(X) > 0$ such that

$$\|f * g\|_{F(X)} \le \kappa \cdot \|f\|_F \cdot \|g\|_{L^1(Y)}$$

whenever $\mathrm{spt} f \subset \mathbb{B}(0,1)$.

(2) Suppose only that convolution maps $F_{\mathrm{cs}} \times L^1_\infty \to F_{\mathrm{loc}}$. Then for each compact $X \subset \mathbb{R}^d$, there exists $\kappa(X) > 0$ such that

$$\|f * g\|_{F(X)} \le \kappa \cdot \|f\|_F \cdot \|g\|_{L^1_\infty}$$

whenever $\mathrm{spt} f \subset \mathbb{B}(0,1)$.

(2) *Suppose all translations are isometries on F, and convolution maps $F \times \mathrm{L}^1 \to F$ (continuously or not). Then*

$$\|f * g\|_F \leq \|f\|_F \cdot \|g\|_{\mathrm{L}^1}.$$

Hence, $* : F \times \mathrm{L}^1 \to F$ is in fact continuous.

Proof. We prove only (3), since the others are similar. For $f \in F, g \in \mathrm{L}^1, h \in F^*$, we have

$$\begin{aligned} |\langle h, f * g\rangle| &= |\int \langle h, \tau_y f\rangle g(y)\, dy| \\ &\leq \|h\|_{F^*} \cdot \|f\|_F \cdot \|g\|_{\mathrm{L}^1}, \end{aligned}$$

and this suffices. ∎

Remark. The point is that as soon as the integrals all exist, $f * g$ is an element of F^{**}, and the map $F \times \mathrm{L}^1 \to F^{**}$ is continuous. If for some reason we know that $f * g \in F$, then we get the estimate, because the injection $F \to F^{**}$ is an isometry. ∎

In the general case we have this:

Proposition 4.4. *Let F be a TSCS, and suppose that convolution maps*

$$F_{\mathrm{cs}} \times \mathrm{L}^1{}_{\mathrm{loc}} \to F_{\mathrm{loc}}.$$

Then the map is continuous.

Proof. Similar to the last. ∎

5. Operators.

Definition. Let S be an operator defined on a set of distributions, with values in the set of distributions.

We say that a topological vector space $F \subset {C^\infty}_{\rm cs}'$ is S–invariant if F lies in the domain of S and S maps F into F, continuously (with respect to the topology of F). This is standard terminology, but we wish to introduce some more, useful when F is an TSCS.

We say that F is locally S–invariant if $S : F \to F_{\rm loc}$, continuously.

We say that F is co–locally S–invariant if $S : F_{\rm cs} \to F$, continuously.

We say that F is bi–locally S–invariant if $S : F_{\rm cs} \to F_{\rm loc}$, continuously.

We make the convention that T–invariant means T_ϕ–invariant, for each $\phi \in {C^\infty}_{\rm cs}$. In fact, invariance under T_ϕ's is actually of interest for a wider class of spaces than the TSCS, so we allow for that in the following definition.

Definition. We say that a LCTVS F of distributions is T–invariant if $T_\phi : F \to F$ continuously, whenever $\phi \in {C^\infty}_{\rm cs}$.

First we discuss invariance under the Cauchy transform, $\mathfrak{C}$.

For general TSCS, F, the $\bar\partial$–problem,

$$\bar\partial u = f \tag{1}$$

with $f \in F$, can rarely be solved with $u \in F$. For instance, it cannot be done in the case $F = {C^\infty}_{\rm cs}$, or more generally, whenever $F = F_{\rm cs}$. This is clear, because in these cases the Cauchy transform necessarily gives the (unique) solution, and there are functions $f \in {C^\infty}_{\rm cs}$ whose Cauchy transform does not have compact support. More generally, it is usually the case that all solutions of (1) behave a little bit worse at ∞ than f does. What saves this situation is that in all reasonable spaces, the Cauchy transform $\mathfrak{C}$ maps $F_{\rm cs}$ into $F_{\rm loc}$, so that (1), with $f \in F_{\rm cs}$, has a solution $u \in F_{\rm loc}$.

Lemma 5.1. *Suppose F is a TSCS that admits a TSCS topology with respect to which it is separable and translation is weakly measurable. Then F is bi–locally C–invariant.*

Proof. By Theorem 4.1, convolution maps $F_{\rm cs} \times {\rm L^1}_{\rm loc}$ into $F_{\rm loc}$, and by Prop. 4.4 the map is continuous with respect to any TSCS topology on F. Since $1/z$ belongs to ${\rm L^1}_{\rm loc}$, that suffices. ∎

Now we relate T–invariance to C–invariance.

Lemma 5.2. *Let F be a TSCS. Then the following are equivalent:*
(1) F is T–invariant,
(2) F is co–locally T–invariant,
(3) F is co–locally C–invariant.

Proof. This is more-or-less obvious, in the light of the formula

$$T_\phi f = \mathfrak{C}\left(\phi \cdot \bar\partial f\right).$$

■

If F is a bi–locally $\mathfrak{C}$–invariant TSCS, then of course F_{loc} is T–invariant, so there are T–invariant TSCS in the local-equivalence class of F. Usually, F_{loc} is a good deal larger than F, and it is desirable to identify smaller T–invariant spaces, locally-equivalent to F. First, we must determine what distinguishes the co–locally $\mathfrak{C}$–invariant TSCS among the bi–locally C–invariant ones. The criterion is very simple.

Lemma 5.3. *Let F be a bi–locally $\mathfrak{C}$–invariant TSCS. Then F is T–invariant if and only if it contains all those distributions $f \in F_{\text{loc}}$ such that f is holomorphic on a full neighbourhood of ∞, and $f(\infty) = 0$.*

Proof. A distribution f is of the form $\mathfrak{C}g$ for some distribution g having compact support, if and only if f is holomorphic on a full neighbourhood of ∞ and $f(\infty) = 0$. Let us denote the set of such distributions by E.

"if": Suppose F contains $F_{\text{loc}} \cap E$. If $f \in F$ and $\phi \in C^{\infty}{}_{\text{cs}}$, then $\mathrm{T}_{\phi} f$ belongs to F_{loc} since F is bi–locally $\mathfrak{C}$–invariant, and belongs to E since $\text{spt}(\phi \cdot \bar{\partial} f)$ has compact support. Hence, it belongs to F. The continuity of the map $\mathrm{T}_{\phi} : F \to F$ follows from Prop. 4.5.

"only if": Suppose F is T–invariant and fix $f \in F_{\text{loc}} \cap E$. Take any $\phi \in C^{\infty}{}_{\text{cs}}$ that equals 1 on a neighbourhood of $\text{spt}\bar{\partial} f$. Then $\mathrm{T}_{\phi} f = \mathfrak{C}\bar{\partial} f = f$, so $f \in F$, as required. ■

Now the space $F_{\text{loc}} \cap E$ of the above proof is not a TSCS. It is not closed under complex conjugation. We can eliminate that problem by replacing the word "analytic" by "harmonic" in the definition of the space E. The resulting space,

$$\{f \in F_{\text{loc}} : f \text{ is harmonic off a compact and tends to 0 at } \infty\},$$

is the least TSCS that contains $F_{\text{loc}} \cap E$ and is thus the least T–invariant TSCS in the local–equivalence class. It is not, however, an SCS, because it lacks full affine–invariance, and the least SCS that contains it is difficult to describe in explicit terms. For applications which require affine–invariance, the situation is saved by the following convenient fact (— recall that F_{∞} is an SCS if F is).

Lemma 5.4. *Let F be a bi–locally $\mathfrak{C}$–invariant TSCS. Then F_{∞} is a T–invariant TSCS.*

Proof. If we take a function $f \in F_{\text{loc}} \cap E$, then for large $|a|$, $f(a + \cdot)$ is analytic on a neighbourhood of $\mathbb{B}(0, 1)$, and hence belongs to $C^{\infty}{}_{\text{cs}}(\mathbb{B}(0, 1))$, and it is easy to see that it tends to 0 in $C^{\infty}{}_{\text{cs}}(\mathbb{B}(0, 1))$. Thus it tends to 0 in $F(\mathbb{B}(0, 1))$, since $C^{\infty}{}_{\text{cs}} \hookrightarrow F$. Thus the condition of Prop.4 is satisfied, so that F_{∞} is T–invariant. ■

The space F_{∞} inherits properties of F such as normability and metrisability. In general, it is neither larger than nor smaller than F. It is an invariant of the local-equivalence class of F. The family of all translations acts equicontinuously on it (— isometrically, in the Banach case). It is presumably a little larger than the minimal T–invariant TSCS in the local-equivalence class, but this is a small price to pay for its other desirable properties. Experience shows that it is, in any case, very little larger than the minimum.

For example, all the spaces L^p for all $p > 2$, C_0, BC^k for all $k \in \mathbb{N}$, Lipβ and lipβ for $\beta > 0$, ZC, ZS, $W^{k,p}$ for all $k \in \mathbb{N}$ and all $p > 1$, are T–invariant TSCBS's (—indeed SCS's). The spaces L^p with $1 \leq p \leq 2$ and $W^{1,1}$ are locally T–invariant, but not T–invariant. When T–invariance is needed, they may be replaced by the (larger) Banach spaces ${L^p}_\infty$ and $W^{1,1}\infty$.

Theorem 5.5. *Suppose F is a TSCS that admits a separable translation–measurable TSCS topology. Then F is locally T–invariant, and F_∞ is T–invariant.*

Proof. Combine the lemmas. ■

We state separately the most useful special cases:

Corollary 5.6. *Let (1) F be a small TSCS or (2) F be the TSCS dual of a small TSCS. Then F is locally T–invariant, and F_∞ is T–invariant.* ■

It is noteworthy that previous proofs of the T–invariance of various special TSCS, such as C, L^p $(p > 2)$, Lipα, and BMO, have involved substantial spadework. This theorem uncovers the essential pattern in these results. The theorem also throws up useful new observations, such as the availability of a T–invariant SCBS that is locally–equivalent to L^2.

The most useful T–invariant spaces that are not TSCS are the spaces ${AC^\infty}_{\rm cs}{}'(U)$ of distributions analytic on a given open set $U \subset \mathbb{S}^2$. Since the intersection of two T–invariant spaces is T–invariant, we obtain the following.

Proposition 5.7. *Let F be a T–invariant space and let $U \subset \mathbb{S}^2$. Then*

$$\mathcal{A}F(U) = \{f \in F : f \text{ is analytic on } U\}$$

is T–invariant.

■

The next proposition provides other ways to extend the list of T–invariant spaces.

Proposition 5.8. *(1) Let E be a T–invariant space and let F be a T–invariant TSCS that contains E. Then the closure of E in F is T–invariant.*
(2) Let $\{E_\alpha\}$ be a net of T–invariant spaces, directed by continuous inclusion maps, and let E be $\bigcup E_\alpha$ with the inductive limit topology. Then E is T–invariant.
(3) Let $\{E_\alpha\}$ be an arbitrary family of T–invariant spaces. Then $\bigcap_\alpha E_\alpha$ is a T–invariant space.

For instance, this can be combined with results above to show that the space of uniform limits on $\mathbb{C}$ of sequences of Lip1 functions that are analytic on a neighbourhood of a certain closed set X, is T–invariant, as is the space of functions approximable in L^4 by functions analytic near a given compact and belonging to some L^p for $p > 8$ (p may depend on the function).

Finally we note that the only properties of the T_ϕ operator used in the above proofs are that the kernel involved,

$$\frac{-1}{\pi z}$$

is locally–integrable, and its translates

$$\frac{-1}{\pi(z-a)}$$

tend to zero in

$$C^\infty(\mathbb{B}(0,1))$$

as $a \to \infty$. Consequently, the results carry over to the T_ϕ^L operator associated to any elliptic operator L that has a parametrix, as long as the parametrix shares these properties. For instance, they work for the Laplacian in d–dimensions if $d > 2$. For general smooth elliptic operators having a parametrix, the behaviour at ∞ will be bad, but we still get the local–integrability, and hence the local T^L–invariance.

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