Time–reversal Symmetries of Dynamical Systems

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Abstract: We consider dynamical systems that admit a time–reversal symmetry, especially in the case when the symmetry is an involution. We give a complete local classification of the one–dimensional real–analytic examples of such systems (near a common fixed point of the system and symmetry), and we observe that they never have stable fixed points. We show that this feature carries over to a very general setting.

1 Introduction

Consider a discrete dynamical system

$$x_{k+1} = \varphi(x_k), \qquad k = 1, 2, 3, \cdots . \tag{1}$$

where $\varphi : X \to X$ is a continuous function on a topological space X. Such systems arise in applied science when a continuous dynamical system is sampled at discrete times, or when its phase–space trajectories cut a Poincaré section. They also arise naturally in finance and computing. From a mathematical point of view, one may regard any function that maps a set into itself as the transition function φ of a dynamical system, and for that reason the results of the theory of discrete dynamical systems may be applied in other areas of mathematics, such as complex analysis and geometry.

Given a system (1), one is interested in the existence of fixed points ($\varphi(x) = x$), periodic points ($\varphi^n(x) = x$), and their character (attracting, repelling, stable, unstable), and in the existence of various forms of chaos. [1]

Poincaré was the first to study such systems in general context [5]. He was motivated by the n–body problem. He himself noticed that systems that admit time–reversal symmetries are special.

Definition 1.1 *A time–reversal symmetry of the dynamical system* (X, φ) *is a homeomorphism* $\sigma : X \to X$ *such that*

$$\sigma^{-1} \circ \varphi \circ \sigma = \varphi^{-1}.$$

Of course, (X, φ) cannot admit such a symmetry unless φ is injective.

For example, let Y be the phase space of a classical n–body system (moving under their mutual gravitational attraction), and let X be the subspace obtained by removing all collision orbits (in forward or backward time). Let $\varphi : X \to X$ be the map that advances the system for one unit of time, and let $\sigma : X \to X$ be the map that fixes each position and reverses each momentum vector. Then σ is a time–reversal symmetry of (X, φ).

As another example, take billiards on an arbitrary smoothly–bounded, convex billiard table, with boundary Γ. Let $X = \Gamma \times (0, \pi)$, and let

$$\varphi(p, \theta) = (p', \theta')$$

where p' is the next point at which the ball strikes Γ, assuming that it leaves Γ at p making an angle of θ with the counterclockwise tangent, and when θ' is the angle at which it leaves p' after striking p'. In this case, the map

$$\sigma : (p, \theta) \mapsto (p, \pi - \theta)$$

is a time–reversal symmetry.

Observe that in both examples

$$\sigma \circ \sigma = \mathbb{1}, \qquad (= \text{the identity map of X})$$

i.e. the symmetry is an involution. Involutions are useful in group theory, and the following lemma is basic, simple and well–known:

Lemma 1.2 *Let* G *be a group and* $\varphi \in G$*. Then the following are equivalent:*
(1) There exists an involution $\sigma \in G$ *with* $\sigma^{-1}\varphi\sigma = \varphi^{-1}$*;*
(2) φ *may be written as the product* $\tau_1\tau_2$ *of two involutions.*

Corollary 1.3 *Let* (X, φ) *be a dynamical system. Then it admits an involutive time–reversal symmetry if and only if* φ *may be written as the composition* $\tau_1 \circ \tau_2$ *of two involutions.*

Motivated by this observation, and by the fact that one of us encountered pairs of non–commuting involutions in diverse contexts, especially in connection with approximation problems [2, 3, 4, 6], we decided to make a systematic study of the phenomenon. In the present paper, we discuss the one–dimensional case.

2 One–dimensional systems

Let $-\infty \le a < c < b \le +\infty$, and $X = (a, b)$, an interval on the real line. There is a simple way to find involutions on X that fix c. Take any smooth (=infinitely–differentiable) function $f : (a, b) \to (0, +\infty)$ with the properties

$$\begin{aligned} &f'(x) < 0, \quad a < x < c, \\ &f'(c) = 0, \\ &f'(x) > 0, \quad c < x < b, \end{aligned} \tag{2}$$

and

$$\lim_{x \downarrow a} f(x) = \lim_{x \uparrow b} f(x). \tag{3}$$

Then, for each $x \in (a, b)$, define $\tau(x)$ by the equation

$$\{z : f(z) = f(x)\} = \{x, \tau(x)\}. \tag{4}$$

In other words, if the level set $f^{-1}f(x)$ has two points, then $\tau(x)$ is the one other than x, and in the other case $(x = c)$, $\tau(x) = x$.

Clearly τ is an involution, and provided f is not *flat* at c (i.e. having all derivatives zero), one can show that τ is smooth, with Taylor expansion

$$\tau(x) \sim c - (x - c) + b_2(x - c)^2 + b_3(x - c)^3 + \cdots$$

about c, and one can express the coefficients b_n in terms of the Taylor coefficients of f at c.

Of course, one can make this construction without assuming that f is differentiable. All one needs is that each set $f^{-1}f(x)$ has no more than two points, and then one may define an involution in the same way. In this way, one obtains involutions on (a, b) that are, for instance, merely continuous, or that are only once or twice differentiable. Some technical difficulties arise if one wishes to study the phenomenon in this generality, so we content ourselves here with the best–behaved case, the case when f is real–analytic. We also assume (without loss in generality) that $c = 0$.

Proposition 2.1 *Let* $f(x) = x^{2m} + a_{2m+1}x^{2m+1} + \cdots$ *be real–analytic near* 0 *and suppose* f *is not even. Then the derived involution* $\tau(x) = -x + b_2x^2 + \cdots$ *is also real–analytic, and the first* n *with* $b_n \neq 0$ *is even.*

Proof: To see the real analyticity, we consider complex x near 0, but nonzero. For such x, the equation

$$f(y) = f(x)$$

has $2m$ solutions, and for small enough x we may unambiguously define $\tau(x)$ as the solution y closest to $-x$. Since $f'(x) \neq 0$, $\tau(x)$ varies holomorphically with x on a deleted neighbourhood of 0, and since it is bounded near 0, the singularity at 0 is removable, and $\tau(x)$ is analytic near 0.

To see the rest, observe that since f is not even, it may be written as

$$f(x) = x^{2m} + \cdots + a_{2n}x^{2n} + a_{2n+1}x^{2n+1} + \cdots$$

with $a_{2n+1} \neq 0$. If $a_{2n+1} > 0$, then $f(x) > f(-x)$ for small positive x, and hence

$$\begin{aligned} \tau(x) < -x \quad &, x > 0 \\ \tau(x) < x \quad &, x < 0, \end{aligned}$$

so that the first n with $b_n \neq 0$ must be even. The case $a_{2n+1} < 0$ is treated similarly.

Corollary 2.2 *If* f_1, f_2 *are two real analytic functions of the above type, and* $\tau_i (i = 1, 2)$ *are the two involutions generated by them, and if* $\tau_1 \neq \tau_2$*, then* 0 *is an unstable neutral point for the dynamical system* $((a, b), \tau_1 \circ \tau_2)$.

Proof: Making a change of variables, one may assume that $\tau_1(x) \equiv -x$. Then the corollary follows easily by graphical analysis [1].

Proposition 2.3 *Locally, all one–dimensional real–analytic involutions arise in the above way. That is, given a real–analytic involution* τ *on an interval of the real line, fixing* $c = 0$*, there exists some real–analytic function* f *defined near* 0*, having the properties (2) and*

$$f(\tau(x)) = f(x)$$

near 0.

Proof: Let $\tau(x) = -x + b_2x^2 + b_3x^3 + \cdots$ be an analytic involution fixing 0. Take

$$f(x) = (x - \tau(x))^2 = x^2 + \cdots .$$

Then $f(\tau(x)) = f(x)$, and f is 2-1 on a punctured neighbourhood of 0, as desired. We immediately deduce:

Proposition 2.4 *Let 0 be a fixed point of a one–dimensional real–analytic dynamical system $((a,b),\varphi)$. Let the involution τ be a time–reversal symmetry of the system, and $\tau(0) = 0$. Then 0 is a neutral unstable fixed point of the system.*

It is clear that the same analysis holds for one–dimensional complex holomorphic dynamical systems, in the neighbourhood of a fixed point.

As for non–analytic systems on the line, much of the analysis goes through as in the real–analytic case, provided f is not flat at the fixed point c. In particular, the above propositions remain valid if "real–analytic" is replaced by "smooth" throughout the statements.

3 Time–reversal symmetry and instability, in general.

The phenomenon observed in Proposition 2.4 occurs quite generally. Let (X,φ) be any discrete dynamical system, and $p \in X$. Recall that p is an *attracting* fixed point if $\varphi(p) = p$ and for each neighbourhood U of p there exists a neighbourhood $V \subset U$ such that $\varphi^n(V) \subset U, \forall\, n \geq 1$, and $\varphi^n(x) \to p$, as $n \uparrow +\infty, \forall\, x \in V$. On the other hand, p is a *repelling* fixed point if φ is bijective near p and p is an attracting fixed point for the system φ^{-1}. In similar ways, attracting and repelling cycles are defined. Attractiveness is a strong kind of stability, and repulsiveness is a strong kind of instability.

It is not normally the case that one and the same point is both attracting and repelling, but it does occur, for instance when X is a discrete space.

Lemma 3.1 *Let X be a T_1 space (i.e. singletons are closed), and let p be an accumulation point of X. Then p cannot be both attracting and repelling for the same system φ on X.*

Proof: Immediate.

Proposition 3.2 *Let X be a T_1 space, and let the accumulation point p be a cyclic point for a system (X,φ), with cycle*

$$C = \{p, \varphi(p), \cdots, \varphi^{n-1}(p)\}.$$

Suppose τ is a time–reversal symmetry of (X, φ), with $\tau(p) \in C$. Then the cycle C is neither attracting nor repelling.

Proof: The symmetry τ conjugates the map φ to its inverse φ^{-1}, and φ^n to φ^{-n}. If C were an attracting cycle for φ, then p would be an attracting fixed point for φ^n, and hence a repelling fixed point for φ^{-n}. But conjugacy preserves the repelling character of a fixed point, so that $\tau(p)$ and hence p are also a repelling fixed points for φ^n. By the lemma, this is impossible. Thus C is not attracting. Similarly (interchanging φ and φ^{-1}), one sees that C is not repelling, either.

4 Examples

A simple example is the familiar fact that a 2–cycle in billiards cannot be attracting.

For an example from linear algebra, consider 2×2 matrices over $\mathbb{C}$, regarding them as self–maps of $\mathbb{C}^2$. Let A,B be two such matrices, with $A^2 = B^2 = \mathbb{1}$. As is well–known, one can make a change of coordinates and conjugate both matrices at once to

$$A = \begin{pmatrix} 0 & \mu \\ \mu^{-1} & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & \mu^{-1} \\ \mu & 0 \end{pmatrix}$$

with $\mu \neq 0$. If the product

$$AB = \begin{pmatrix} \mu^2 & 0 \\ 0 & \mu^{-2} \end{pmatrix}$$

is not an isometry, then it expands in one direction and contracts in another.

More generally, if X is a Riemannian manifold and $\varphi : X \to X$ is differentiable with a fixed point p and a time–reversal symmetry τ that fixes p, then $\det(d\varphi) = \pm 1$, and it follows that expansion in some directions must be balanced by compression in others.

As a global example, consider $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as the sphere $\mathbb{S}^2$, and let

$$f(z) = \frac{az^2 + bz + c}{dz^2 + ez + f}$$

be a quadratic rational function. Define an involution $\tau : S^2 \to \mathbb{S}^2$ by

$$f^{-1}(f(z)) = \{z, \tau(z)\}.$$

Then τ is an analytic function, and injective, and hence

$$\tau(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

Each involutive Möbius transformation arises in this way: given an involution

$$\tau(z) = \frac{\alpha z + \beta}{\gamma z + \delta},$$

take (cf. the proof of (2.3))

$$f(z) = (z - \tau(z))^2 = \left\{ \frac{(\gamma - \alpha)z + (\delta - \beta)}{\gamma z + \delta} \right\}^2 .$$

Given two f's, say f_1 and f_2, one obtains two involutions τ_1 and τ_2, and a system $\varphi = \tau_1 \circ \tau_2$ admitting a time–reversal symmetry τ_1. Of course, φ is also a Möbius transformation. There are three cases:

1°. There is just one fixed point, p of φ. Then $\tau_1(p) = \tau_2(p) = p$ and p is a neutral point of φ.

2°. There are two fixed points p_1, p_2 of φ, and $\tau_1(p_1) = p_1$. Then $\tau_1(p_2) = p_2$ and (by (3.2)) both points are neutral, and φ is an elliptic transformation.

3°. These are two fixed points p_1, p_2 of φ, and $\tau_1(p_1) = p_2$. Then $\tau_1(p_2) = p_1$, and (3.2) does not prevent p_1 being an attractive fixed point for φ (as long as p_2 is repelling, with reciprocal multiplier). This case actually occurs. Take, for instance,

$$\varphi(z) = 2z,$$
$$\tau_1(z) = -\frac{1}{z}.$$

This example shows that the hypothesis $\tau(p) \in C$ is necessary in (3.2).

Remark: It is well known that for systems obtained by discretising a classical Hamiltonian system, such as the n–body system mentioned above, the principle of conservation of density–in–phase implies that no periodic point can be attrcting or repelling.

Remark: Involutive time–reversal symmetries have other uses. A group generated by two involutions is called *dihedral*, because the group of symmetries of a dihedral crystal is of this type. Finite or infinite, such a group has a cyclic subgroup of index 2, and is "small" in the sense that the number of elements expressible using words of length n in the generators grows linearly with n. As a result, tools such as ergodic theory, mean–value operators, covariant measures and the like are available. Such tools were used, for instance, in [2, 3].

Remark: Finally, we note that the existence of time–reversal symmetries is only interesting in categories finer than the topological. In the category of sets, each bijection factors as the composition of two involutions, as may be readily verified by considering the action of the bijection on its orbits (where it is isomorphic to the action of a generator on some finite or infinite cyclic group). The subject begins to present challenges when one requires at least continuity of the factors. There may be trivial obstacles, such as the nonexistence of any (non–identity) involutions on the topological space X. So a natural question would be to ask for a classification of the dynamical systems admitting an involutive time–reversal symmetry, under the assumption that the involutions generate the automophism group of X, or some other assumption that guarantees a reasonable supply of involutions.

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